Log Angles: Characteristic Angles of Crystal Orientation Given by the Logarithm of Rotation Matrix

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A rotation matrix \( R \) with respect to a reference frame is used to describe certain crystal orientation. The logarithm of \( R \), \( \ln R \) is a skew symmetric tensor with three independent elements of real numbers. The goniometer-stage model in the present study shows that the three independent elements of \( \ln R \) are the characteristic angles of \( R \) representing the rotation angles around coordinate axes. Different from various kinds of the Euler angles, the characteristic angles called the log angles are uniquely determined for certain \( R \).

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1. Introduction

Crystal orientations and their changes are important factors to investigate material microstructures.1–3) For instance, to investigate changes of microstructures by rolling and plastic-deformation mechanisms during rolling, crystal orientations of plate-like single crystals before and after rolling are compared.4–6) Depending on the initial orientations or rolling conditions, significant changes of crystal orientations may occur. However, the single crystalline states are sometimes maintained even after the changes of crystal orientations. In such cases, the orientation changes are often discussed by using the reference frame of rolling (RD: rolling direction, TD: transverse direction, ND: normal direction).4–6)

When indicating a crystal orientation with respect to a reference frame, we can make geometrically sound discussion by using a three-dimensional orthogonal rotation matrix \( R \). Although the number of independent elements of the 3x3 matrix \( R \) is three, \( R \) generally has nine different elements. Because of the complexity of the relationships among the elements of \( R \), intuitive understanding of the changes of crystal orientations from changes of the elements of \( R \) is not easy.

To grasp the meaning of crystal orientation described by \( R \), the Euler angles of \( R \) consisting of a set of three angles are often used. The rotation matrix \( R \) can be decomposed into the product of three successive rotations around the \( x \), \( y \) or \( z \) coordinate axes of the frame. The Euler angles are the three rotation angles around the coordinate axes.7) However, the values of the three rotation angles for certain \( R \) change depending on the order of the three successive rotations. This shows that various sets could exist for the Euler angles and the three angles cannot be determined uniquely for certain \( R \).7) For example, when we would like to consider a change of crystal orientation of single crystal caused by rolling and evaluate a ratio of the rotation around TD in the whole rotation8,6,8), the ratio cannot be uniquely determined from the concept of the Euler angles.

Similar to the exponential and logarithmic functions of number, the exponential and logarithmic functions of matrix can be defined by the sum of series.9–11) It is known that the logarithm \( \ln R \) of \( R \) is a skew symmetric tensor with three independent elements of real numbers.9–11) In this study, we will show that the three elements of \( \ln R \) are the set of three characteristic angles of \( R \). The characteristic angles called the log angles give the intuitive understanding of the crystal orientation described by \( R \). Different from the various Euler angles, the log angles are uniquely determined for certain \( R \). As an example to compare the log angles with the Euler angles, we will discuss previous experimental results12,13) on the change of crystal orientation. As an application of the log angles, we will analyze our recent results on the changes of crystal orientations observed in a single crystal after rolling.

2. Analysis

2.1 Combination of the rotations around the coordinate axes

When \( R_x \), \( R_y \), and \( R_z \) are respectively the rotation matrices around the \( x \), \( y \) and \( z \) coordinate axes of the orthogonal frame, the elements of these matrices are written as

\[ R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix}, \]

(1-a)

\[ R_y(\varphi) = \begin{pmatrix} \cos \varphi & 0 & \sin \varphi \\ 0 & 1 & 0 \\ -\sin \varphi & 0 & \cos \varphi \end{pmatrix}, \]

(1-b)

and

\[ R_z(\psi) = \begin{pmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \]

(1-c)

where \( \theta \), \( \varphi \) and \( \psi \) are respectively the rotation angles around the \( x \), \( y \) and \( z \) coordinate axes.

We can represent any \( R \) with the product of such three ba-
rotation matrices. Here we consider \( R \) consisting of the product of \( R_3 \) with \( \theta_1 \), \( R_2 \) with \( \varphi_1 \) and \( R_z \) with \( \psi_1 \) in this order:

\[
R = R_z(\psi_1)R_x(\varphi_1)R_y(\theta_1). \tag{2}
\]

Figure 1 is a goniometer-stage model, which is convenient to understand the combination of rotations. In this model, a goniometer stage consists of three parts connected in series and has the function of triaxial rotations. The goniometer stage and the reference frame with the \( x \), \( y \) and \( z \) coordinate axes are fixed on a base plate. A crystal with the \( x' \), \( y' \) and \( z' \) coordinate axes is located on the top of the goniometer stage. When all of \( \theta_1 \), \( \varphi_1 \) and \( \psi_1 \) are zero, the primed axes are parallel to the unprimed reference axes. The rotation angles of the three parts determine the directions of the primed axes of the crystal. The directions of the primed axes using the reference \( x'\)-\( y'\)-\( z' \) frame are given by \( R \) as

\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}
\]

The goniometer-stage model in Fig. 1 is a graphical or mechanical representation of eq. (2). This is explained in the next section.

When a direction is denoted by a vector \( v \), the direction after the rotation as much as \( R \) is given by \( Rv \). When \( R \) is the product of matrices given by eq. (2), the rotation is interpreted as the three successive rotations with respect to the reference frame in the order of \( R_3(\psi_1) \), \( R_2(\varphi_1) \) and \( R_z(\theta_1) \). The goniometer stage shown in Fig. 1 consists of three parts connected in series. The operation of each part from top to bottom in Fig. 1 corresponds to the operations of \( R_3 \), \( R_2 \) and \( R_z \) in this order. The operation of the top part does not change the original rotation axes of the other two lower parts. Hence the following successive operations of the parts

1. The top part, \( R_3 \) as much as \( \psi_1 \)
2. The middle part, \( R_2 \) as much as \( \varphi_1 \)
3. The bottom part, \( R_z \) as much as \( \theta_1 \)

in this order are the same as the successive rotations around the coordinate axes given by the right-hand side of eq. (2).

The angles \( (\theta_1, \varphi_1, \psi_1) \) shown graphically in Fig. 1 are the Euler angles of \( R \) represented by eq. (2). In textbooks explaining the Euler angles, it is often emphasized that, in successive rotations, the first rotation around a certain coordinate axis changes the orientation of rotation axes of the subsequent rotation. From Fig. 1, we can understand that this is also true. If we operate the three parts in Fig. 1 from bottom to top, the operation of the bottom part corresponding to \( R_z \) changes the rotation axis of the middle part from the \( y \)-axis. The order of the operations of the three parts affects intermediate states of the whole rotation. However, the final rotation, of course, becomes identical as far as each rotation angle of the three parts in Fig. 1 is the same.

The Bunge Euler angles, or the Euler angles due to Bunge, is often used for the texture analysis. Different from the goniometer-stage model shown in Fig. 1, the model for the Bunge Euler angles consists of two top and bottom parts corresponding to \( R_3 \) and one middle part corresponding to \( R_z \). On the other hand, the model for the Euler angles explained in a usual textbook on linear algebra consists of two top and bottom parts corresponding to \( R_3 \) and \( R_z \), and one middle part corresponding to \( R_x \). Various kinds of the Euler angles hence correspond to their goniometer-stage models consisting of appropriate combinations of the three parts. However, the components of the rotation angles or the unique characteristic angles around the coordinate axes cannot be represented by any set of the Euler angles. This is because when the order of three parts of the goniometer-stage changes, the rotation angles around the coordinate axes change for the same \( R \). In other words, matrix multiplication is not commutative. When the order of the matrix multiplication is changed, the product of the matrices is not the same generally. For example, when the order changes, we generally have

\[
R_z(\theta_1)R_2(\varphi_1)R_x(\psi_1) \neq R_x(\psi_1)R_2(\varphi_1)R_z(\theta_1) \tag{4}
\]

corresponding to a certain set of \( (\theta_1, \varphi_1, \psi_1) \). On the other hand, when the products of different orders are the same:

\[
R_x(\psi_1)R_2(\varphi_1)R_z(\theta_1) = R_z(\theta_2)R_2(\varphi_2)R_x(\psi_2) \tag{5}
\]

we generally have

\[
\theta_1 \neq \theta_2, \varphi_1 \neq \varphi_2, \psi_1 \neq \psi_2. \tag{6}
\]

However, in the case of infinitesimal rotation neglecting infinitesimals of second order, the multiplication of rotation matrices can be treated to be commutative. The product of rotation matrices for infinitesimal rotations will be shown later.

### 2.2 Logarithm ln R of rotation matrix \( R \)

For the two-dimensional rotation matrix \( R_{2D} \) written as

\[
R_{2D} = \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}, \tag{7}
\]

its logarithm \( \ln R_{2D} \) is a skew symmetric tensor with an element of rotation angle \( \theta \):

\[
\ln R_{2D} = \begin{pmatrix}
0 & -\theta \\
\theta & 0
\end{pmatrix}. \tag{8}
\]

To consider the logarithm of the three-dimensional rotation matrix \( R \), it is convenient to express \( R \) by the axis/angle pair.
The axis/angle pair consists of the unit vector $\mathbf{V} = (h, k, l)$ showing the rotation axis and the rotation angle $\phi$ around $\mathbf{V}$. Using $(h, k, l)$ and $\phi$, the elements of $\mathbf{R}$ are written as:

$$
\mathbf{R} = \begin{pmatrix}
(1 - h^2) \cos \phi + h^2 & hh(1 - \cos \phi) + l \sin \phi & lh(1 - \cos \phi) - k \sin \phi \\
hk(1 - \cos \phi) + l \sin \phi & (1 - k^2) \cos \phi + k^2 & kl(1 - \cos \phi) + h \sin \phi \\
lh(1 - \cos \phi) - k \sin \phi & kl(1 - \cos \phi) + h \sin \phi & (1 - l^2) \cos \phi + l^2
\end{pmatrix}
$$

(9)

The logarithm of $\mathbf{R}$ expressed by eq. (9) is also a skew symmetric tensor written as:

$$
\ln \mathbf{R} = \begin{pmatrix}
0 & -l\Phi & k\Phi \\
l\Phi & 0 & -h\Phi \\
-k\Phi & h\Phi & 0
\end{pmatrix}.
$$

(10)

These three independent elements $h\Phi$, $k\Phi$, and $l\Phi$ of $\ln \mathbf{R}$ are called the log angles of $\mathbf{R}$ in the present paper.

The logarithm of matrix has been considered when we have derived the Hencky strains from stretch tensors.\(^{16–19}\)

The procedure to obtain the logarithm of matrix in this case is as follows.\(^{16}\)

(a) Diagonalization of stretch tensors by rotational transformation of coordinates from original coordinate systems. Since the stretch tensors are positive definite symmetric matrices, diagonal elements of the matrices after diagonalization are always positive.

(b) Calculation of the logarithms of the diagonalized matrices. For the diagonalized matrices with positive elements, their logarithms are obtained by calculating the logarithms of the positive diagonal elements.

(c) Inverse rotational transformation of the logarithms of the matrices to the original coordinate system.

The procedure to obtain the logarithms of the rotation matrices is essentially the same as the above one.\(^{10,11}\) However, complex numbers temporary appear during the calculations to obtain the logarithms.\(^{10,11}\) Considering the meaning of logarithm, eqs. (8) and (10) can be derived by a procedure different from the above one. This is shown in Appendix.

2.3 Interpretation of the log angles using a goniometer-stage model

The relationship between $\mathbf{R}$ and $\ln \mathbf{R}$ is given by:

$$
\mathbf{R} = \lim_{p \to \infty} \left( \mathbf{E} + \frac{\ln \mathbf{R}}{p} \right)^p,
$$

(11)

where $\mathbf{E}$ is a unit matrix. For a real number $\lambda$, we have the relationship given by:

$$
e^\lambda = \lim_{p \to \infty} \left( 1 + \frac{\lambda}{p} \right)^p.
$$

(12)

Using $\eta = e^\lambda$ which is positive, the relationship $\eta$ and its logarithm $\ln \eta$ is written as:

$$
\eta = \lim_{p \to \infty} \left( 1 + \frac{\ln \eta}{p} \right)^p.
$$

(13)

Equation (11) for matrix is an extension of eq. (13) for number.\(^{9}\)

When $N$ is a sufficiently large positive integer, from eqs. (10) and (11), we have:

$$
\delta \mathbf{R}_S = \mathbf{E} + \frac{\ln \mathbf{R}}{N} = \begin{pmatrix}
1 & -h\Phi/N & k\Phi/N \\
-h\Phi/N & 1 & -h\Phi/N \\
k\Phi/N & h\Phi/N & 1
\end{pmatrix}
$$

(14)

and

$$
\mathbf{R} \approx (\delta \mathbf{R}_S)^N.
$$

(15)

This relationship means that $\mathbf{R}$ is equivalent to the $N$ times operations of $\delta \mathbf{R}_S$. The infinitesimal rotation $\delta \mathbf{R}_S$ written as eq. (14) is actually obtained when we replace $\Phi$ in eq. (9) with $\Phi/N(\ll 1)$ and use the approximations $\sin(\Phi/N) \approx \Phi/N$ and $\cos(\Phi/N) \approx 1$. Considering the infinitesimal angles

$$
\delta \theta = h\Phi/N, \delta \phi = k\Phi/N \text{ and } \delta \psi = l\Phi/N
$$

and eqs. (1-a) to (1-c), we have the relationship written as:

$$
\delta \mathbf{R}_S \approx \mathbf{R}_s(h\Phi/N)\mathbf{R}_s(k\Phi/N)\mathbf{R}_s(l\Phi/N).
$$

(17)

In the case of infinitesimal rotations neglecting infinitesimals of second order, the multiplication of rotation matrices can be treated to be commutative. Equation (17) is satisfied even if the order $\mathbf{R}_s, \mathbf{R}_s, \mathbf{R}_s$ of the right-hand side is changed.

Figure 2 shows the goniometer-stage model for $\mathbf{R}$ given by eqs. (14) to (17). In Fig. 2, a single unit in a spherical body consists of three parts in series that make infinitesimal rotations as much as $h\Phi/N$, $k\Phi/N$ and $l\Phi/N$ respectively around the $x$, $y$, and $z$ axes. The goniometer stage consists of the units stacked $N(\gg 1)$ times as shown in Fig. 2. From Fig. 2, we find that the log angles $h\Phi$, $k\Phi$ and $l\Phi$ are respectively the sums of the divided rotation angles around the $x$, $y$, and $z$ coordinate axes. In the case of the goniometer-stage model consisting of the three parts as shown in Fig. 1, the order of the parts affects the rotation angles around the coordinate axes. However, the log angles $h\Phi$, $k\Phi$ and $l\Phi$ are determined uniquely for certain
The log angles hence are considered to be the characteristic angles of $R$.

When the rotation axis given by $V$ is close to a certain coordinate axis, the component of the rotation angle or the log angle around the axis becomes naturally large. The division and addition of the rotations could also be interpreted as the simultaneous rotations as much as the log angles around the coordinate axes.

## 3. Comparison of the Log Angles with the Euler Angles

As an example to consider the log angles, we analyze the orientation change of a single crystal by rolling. Recently, Yoshida et al.\textsuperscript{12,13} have reported the crystal orientation change of a copper single crystal caused by cold rolling. Figures 3 are stereographic projections showing (a) the initial orientation and (b) the average orientation\textsuperscript{13} after 50% rolling. These are shown using the reference frame of rolling: $x'//\text{ND}$, $y'//\text{TD}$ and $z'//RD$. The $<100>$ directions of the initial crystal orientation $x$, $y$ and $z$ shown in (a) respectively changed to $x'$, $y'$ and $z'$ shown in (b) after rolling.

Using the definition of the rotation matrix given by eq. (3), $R_i$ giving the relationship between the $x$-$y$-$z$ reference frame and the $x'$-$y'$-$z'$ frame before rolling is written as

$$R_i = \begin{pmatrix} 0.598 & 0.252 & 0.761 \\ -0.372 & 0.928 & -0.016 \\ -0.710 & -0.273 & 0.649 \end{pmatrix}. \quad (18)$$

On the other hand, $R_f$, giving the relationship between the $x$-$y$-$z$ reference frame and the $x''$-$y''$-$z''$ frame after rolling is written as

$$R_f = \begin{pmatrix} 0.424 & 0.349 & 0.836 \\ -0.686 & 0.726 & 0.046 \\ -0.591 & -0.593 & 0.547 \end{pmatrix}. \quad (19)$$

In this case, using the $x$-$y$-$z$ reference frame, the matrix $R_{i,f}$ giving the rotation caused by rolling is given by

$$R_{i,f} = R_i R_i^{-1} = \begin{pmatrix} 0.977 & 0.153 & 0.146 \\ -0.193 & 0.928 & 0.318 \\ -0.087 & -0.339 & 0.937 \end{pmatrix}. \quad (20)$$

Numerical calculations to obtain the logarithm of a matrix can be easily performed by recent computation programs. We can also obtain the logarithm of $R$ from eq. (10) by using the values of the axis/angle pair $V/\Phi$, which are calculated easily.\textsuperscript{19} Calculating the elements of the logarithm in $R_{i,f}$ of $R_{i,f}$ given by eq. (20), the log angles of $R_{i,f}$ are obtained as

$$h\Phi = -0.337, \quad k\Phi = 0.119 \quad \text{and} \quad l\Phi = -0.177. \quad (21)$$

Here we compare the log angles of eq. (21) with the Euler angles of $R_{i,f}$. In addition to the Euler angles given by eq. (2), we consider two other sets of the Euler angles given by changing the order of the successive rotations. The two sets are for the successive rotations given by

$$R = R_i(\varphi_2)R_i(\varphi_3)R_i(\theta_2), \quad (22-a)$$

and

$$R = R_i(\varphi_3)R_i(\theta_3)R_i(\varphi_2). \quad (22-b)$$

Figure 4 shows the comparison of the log angles ($h\Phi, k\Phi, l\Phi$) given by eq. (21) with the three sets of the Euler angles: $(\theta_1, \varphi_1, \psi_1)$ given by eq. (2), $(\theta_2, \varphi_2, \psi_2)$ given by eq. (22-a) and $(\theta_3, \varphi_3, \psi_3)$ given by eq. (22-b). Figures 4(a) to (c) show the rotation angles around the $x$, $y$ and $z$-axes respectively. As shown by these figures, the value of the log angle around each axis is different from those of the Euler angles but close to the averages of the Euler angles. The magnitude of the absolute values of the log angles is $|h\Phi| > |k\Phi| > |l\Phi|$ for $R_{i,f}$ of eq. (20).

In this case, the relative differences between the log angles and the Euler angles become larger as the absolute values of the log angles become smaller. As described in 2.1, the Bunge Euler angles are those for the successive rotations around the
z, x and z-axes. In this case, the values of the Euler angles are quite different from those of the log angles. When we would like to consider the decomposed rotation angles around the three coordinate axes, the log angles are the most reasonable measure because of their uniqueness.

4. Application of the Log Angles: Rotation Angle around TD in Orientation Change Caused by Rolling

Significant rotations around TD often occur in rolled single crystals. To consider this phenomenon, we have observed the change in crystal orientation by rolling for a copper single crystal of 99.9 mass% purity. A plate-like specimen with a 2 mm thickness, 20 mm width and length having the initial orientation of [111]///ND, [110]///TD and [112]///RD has been cold rolled with lubrication at room temperature. As an example of the orientation change after rolling, the [111]-pole figures before and after 15% reduction are shown in Fig. 5. The white triangles in the Fig. 5 show the [111] poles of the crystal before rolling and the black dots show those after rolling. The orientation measurements after rolling were made by an SEM-EBSD method for a region located in the center of the specimen. The black dots in Fig. 5 are the results of measurements at 850 points on a line of about 1500 μm length along RD.

From Fig. 5, we find qualitatively that the change of the crystal orientation after rolling is close to the rotation around TD. This rotation around TD can be evaluated quantitatively by the log angles. The results are shown in Figs. 6 (a) and (b). In Fig. 6 (a), the horizontal and vertical axes show the distance along a line parallel to RD and the variations of the log angles $\omega_{TD}$ around TD respectively. In Fig. 6 (b), the horizontal axis is the same as that in Fig. 6 (a), but the vertical axis shows the ratio of $\omega_{TD}/\Phi$. As shown in Fig. 6 (b), the crystal orientation change in this case satisfies $\omega_{TD}/\Phi > 0.9$ at most of the measurement points. However, we can also find that there are some points where $\omega_{TD}/\Phi$ is small. These are important information to discuss arrays of dislocations and deformation structures after rolling. We will consider such experimental results using the log angles and discuss the mechanism of deformation structures.

5. Conclusions

The three-dimensional orthogonal rotation matrix $R$ is important to consider a crystal orientation with respect to a reference frame. The logarithm of $R$, ln$R$ is a skew symmetric tensor with three independent elements of real numbers. In the present study, we have shown by the goniometer-stage model that the three independent elements of ln$R$ called the log angles are the sums of the divided rotation angles around the coordinate axes. Different from the Euler angles, the log angles are uniquely determined for certain $R$ and considered to be the characteristic angles of $R$.

REFERENCES

Appendix

A1: Change in the logarithm ln\(\lambda_p\) of scalar \(\lambda_p\)

We first consider the change in the scalar \(\lambda_p\) as a function of the time \(t\) assuming that \(\lambda_p\) is always positive and continuous. When the change in \(\lambda_p\) during the infinitesimal time interval \(dt\) is \(\delta\lambda_p\), the rate \(v_\lambda\) is given by

\[
 v_\lambda = d\lambda_p/dt \approx \delta\lambda_p/dt. \tag{A1}
\]

On the other hand, when we are interested in the normalized change \(\delta\lambda_p/\lambda_p\) instead of the change \(\delta\lambda_p\) itself, it is necessary to consider the rate \(v_n\) given by

\[
 v_n = (d\lambda_p/dt)/\lambda_p = v_\lambda/\lambda_p.
\]

As shown by the following equation, the rate \(v_n\) can be regarded as the rate of the logarithm \(\ln\lambda_p\):

\[
 d(\ln\lambda_p)/dt = d(\ln\lambda_p)/d\lambda_p \cdot d\lambda_p/dt = \lambda_p^{-1}d\lambda_p/dt = v_\lambda = v_n. \tag{A2}
\]

Hence, in order to consider the variation in the normalized \(\lambda_p\), the variation in the logarithm \(\ln\lambda_p\) should be discussed.

A2: Logarithms of rotation matrices

We next consider the logarithms of rotation matrices. When the rotation angle is \(\theta\), the two-dimensional rotation matrix \(R_{2D}\) is written as

\[
 R_{2D} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}. \tag{A3}
\]

When the rotation angle \(\theta\) changes continuously as a function of the time \(t\), the rate of \(R_{2D}\) is given by differentiating the elements of \(R_{2D}\) with respect to \(t\):

\[
 dR_{2D}/dt = d\theta/dt \begin{pmatrix} -\sin \theta & -\cos \theta \\ \cos \theta & -\sin \theta \end{pmatrix}. \tag{A4}
\]

For the scalar \(\lambda_p\), the reciprocal \(\lambda_p^{-1}\) appears in calculations to obtain the variation in \(\ln\lambda_p\), as shown in the third-side of eq. (A2). In the case of matrix, its reciprocal corresponds to the inverse matrix. Hence, for the matrix \(R_{2D}\), by using the inverse matrix \(R_{2D}^{-1}\) written as

\[
 R_{2D}^{-1} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \quad \tag{A5}
\]

and eq. (A4), the matrix version of eq. (A2) is written as

\[
 R_{2D}^{-1} dR_{2D}/dt = dR_{2D}/dt R_{2D}^{-1} = \frac{d}{dt} \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}. \tag{A6}
\]

The two matrices \(R_{2D}^{-1}\) and \(dR_{2D}/dt\) are commutative as shown by calculating the products. The third side of eq. (A6) is the time derivative of the logarithm \(\ln R_{2D}\) given by eq. (A7). This means that the first and second sides of eq. (A6) give the variation in \(\ln R_{2D}\). These sides have a form essentially the same as that of the third side of eq. (A2) for the scalar \(\lambda_p\).

The logarithm of the three-dimensional rotation matrix \(R\) is also given in a similar manner. Using eq. (9), the inverse matrix of \(R\) is written as

\[
 R^{-1} = \begin{pmatrix} (1-h^2) \cos \Phi + h^2 & hk(1-\cos \Phi) + l \sin \Phi \\ hk(1-\cos \Phi) - l \sin \Phi & (1-k^2) \cos \Phi + k^2 \end{pmatrix} \begin{pmatrix} l \cos \Phi + k \sin \Phi \\ kl \sin \Phi + h \cos \Phi \end{pmatrix}, \tag{A7}
\]

When \(\Phi\) changes as a function of \(t\) for fixed \(V = (h, k, l)\), \(dR/\) \(dt\) is given by

\[
 \frac{dR}{dt} = \frac{d\Phi}{dt} \begin{pmatrix} -(1-h^2) \sin \Phi & hk \sin \Phi - l \cos \Phi \\ hk \sin \Phi + l \cos \Phi & -(1-k^2) \sin \Phi \\ lh \sin \Phi - k \cos \Phi & kl \sin \Phi + h \cos \Phi \end{pmatrix} \begin{pmatrix} -l \sin \Phi + k \cos \Phi \\ k \cos \Phi - l \sin \Phi \end{pmatrix}. \tag{A8}
\]

From eqs. (A7), (A8) and (10), we have the following equation which is essentially the same as eq. (A6):

\[
 \frac{dR}{dt} R^{-1} = R^{-1} \frac{dR}{dt} = \frac{d\Phi}{dt} \begin{pmatrix} 0 & -l & k \\ l & 0 & -h \\ -k & h & 0 \end{pmatrix} = \frac{d(\ln R)}{dt}. \tag{A9}
\]