A New Theory of Fatigue Founded on the Hypothesis that the Distribution of Internal Stresses Varies with Fatigue. (Part III) Alternating Tension and Compression of a Round Bar.

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(Received May 12,1952)

Synopsis

The chief difference from the theories in the first and second reports is that the condition of the total moment maintained constant is replaced by the condition of the total stress maintained constant. In the case of a round bar where the stresses in the interior are smaller than those on the periphery, the following formulae are obtained:

$$\sigma=\sigma_W+C(1/N_B)^{1/2},\quad \gtrsim (2/3)H_1\leqq\sigma_W<H_1;\Delta W=\text{const.}$$

When the stresses in the interior are larger than those on the periphery, we have

$$\bar{\sigma}=\sigma_W+C(1/N_B)^{1/2},\qquad H_1/3\lesssim\sigma_W<H_1,\quad C=\{(3/2)E\Delta W/\Phi\}^{1/2};\Delta W=\text{const.}$$

In the special case of almost homogeneous distribution of strains, the fatigue strength σ_W for completely reversed cycles of longitudinal stress tends toward H_1 which is equal to the fatigue strength for completely reversed bending stress, in our idealized models, in agreement with experiments(1)(2).

I. Model A

We can imagine a case where we have homogeneous distribution of stresses, but in reality such a case can rarely happen even in the idealized case, and the stress as well as the strain varies in degrees in places. First, let us consider a model in which the stresses distribute concentrically in a cross-section of the test-piece, the minimum value being on the axis. This model will be called "Model A" in this paper. When the distribution of tensile stresses and of compressive stresses are unsymmetrical, the fracture would occur first in the weaker side, but in this paper, I treat the case when both sides are symmetrical to each other. I suppose the following relations by the aid of **Fig. 1**:

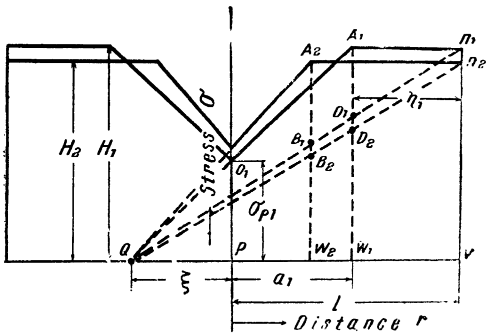

Fig. 1 Model A.

$$\mathcal{E}=cr+\mathcal{E}_p,\ \text{for}\ 0\leqq r\leqq L, \qquad (1)$$

$$\mathcal{E}_p=\text{const.},\ c=\text{const.}, \qquad (2)$$

$$d\mathcal{E}=c\cdot dr,\quad c=\text{const.}, \qquad (3)$$

$$\text{OA: } \sigma=Kr+\sigma_p,\ \text{for}\ 0\leqq r\leqq a, \qquad (4)$$

where σ_p may vary with fatigue, and An: $\sigma=H$, for $a\leqq r\leqq L$, ...(5)

and we get

$$K=cH/(ac+\mathcal{E}_p). \qquad (6)$$

We can easily assume

$$\sigma=(K/c)\mathcal{E},\ \text{for}\ r=0,\ 0\leqq\mathcal{E}\leqq\mathcal{E}_p. \qquad (7)$$

K and c are linear functions of $\mathcal{E}$ only in (7), but there K/c is constant. From(6) we have for the increment from fatigue,

$$\Delta K=(K/H)\,\Delta H-(K^2/H)\,\Delta a. \qquad (8)$$

The variation with fatigue occurs in such a way that, as in **Fig.1**, the elastic straight line QA rotates around the fixed point Q which is formally on the position

$$r=-\mathcal{E}_p/c \qquad (9)$$

(1) R. Nagai, Nippon Kinzoku Gakkai Kôen-Gaiyô, April, 1950.
(2) R. Nagai & M. Isikawa, Nippon Kinzoku Gakkai Kôen-Gaiyô, October, 1951.

according to (1). The line QO is imaginary. In Fig.1 the total strain energy is given by

$$E=\int_0^a 2\pi r\left\{\int_0^{\epsilon_p}\sigma d\varepsilon+\int_{\epsilon_r}^{\epsilon_p+cr}\sigma d\varepsilon\right\}dr+\int_a^L 2\pi r\left\{\int_0^{\epsilon}\sigma d\varepsilon+\int_{\epsilon_p}^{\epsilon_p+cr}\sigma d\varepsilon+\int_{\epsilon_p+cr}^{\epsilon_p+r}Hd\varepsilon\right\}dr, \quad \cdots\cdots(10)$$

so that taking (7) etc. into consideration,

$$\Delta E=\{\pi cKL^4/(4H)+2\pi\varepsilon_pKL^3/(3H)+\pi\varepsilon_p^2KL^2/(2cH)\}\ \Delta H+\{\pi cK^2L^4/(4H)$$
$$+2\pi\varepsilon_pK^2L^3(3H)+\pi\varepsilon_pK^2L^2/(2cH)-2\pi c\ KL^2\eta\}\Delta\eta. \cdots\cdots(11)$$

We have the total stress P as follows:

$$P=2\pi Ka^3/3+\pi K\varepsilon_p\eta^2/c+\pi H(L^2-a^2). \cdots\cdots(12)$$

From $\Delta P=0$, ……(13)

neglecting the terms higher than η, we have the following eq. as the condition of the total stress maintained constant:

$$\Delta H=[-K+6cHH_s\eta/\{KL^2(2cL+3\varepsilon_p)\}]\Delta\eta, \cdots\cdots(14)$$

where $H_s=-K\varepsilon_p/c+H$. ……(15)

From (11) (14) and the relation

$$H-KL-(K/c)\varepsilon_p=-\eta K \cdots\cdots(16)$$

which is obtained from Fig.1, we have finally

$$\Delta E=\pi H_s\ (3c^2L^2+8cL\varepsilon_p+6\varepsilon_p^2)\eta\Delta\eta/\{2(2cL+3\varepsilon_p)\}. \cdots\cdots(17)$$

Denoting the mean stress per unit area of a cross-section perpendicular to the axis by $\bar{\sigma}$, we have $P=\bar{\sigma}\pi L^2$. ……(18)

Putting (12)(18) equal to each other, and neglecting the terms of η^2 and taking (16) into consideration, we have finally

$$\bar{\sigma}=(2/3)H_1+(1/3)\sigma_p+(2/3)\ K_1\eta. \cdots\cdots(19)$$

Integrating the result after substituting "d" for "Δ" in (17), we obtain

$$E_u=\pi H_s(3c^2L^2+8\varepsilon_pcL+6\varepsilon_p^2)\eta^2/\{4\ (2cL+3\varepsilon_p)\}. \cdots\cdots(20)$$

This is the effective energy of the idealized curve. The increase of the strain energy with fatigue is expressed by

$$E_t=\pi H_s(3c^2L^2+8\varepsilon_pcL+6\varepsilon_p^2)(\eta_2^2-\eta_1^2)/\{4(2\ cL+3\varepsilon_p)\}. \cdots\cdots(21)$$

The strain energy increased per one cycle is $(\Delta E)_1=\Phi E_u$, ……(22)

where $\eta_1\leqq\eta\leqq\eta_2,\ \eta_2-\eta_1\ll\eta<L$. ……(23)

The number of repetition before fracture is $N_B=E_t/\Delta E)_1$. ……(24)

Applying (20)(21)(22) to (24), we have $\eta=\{(\eta_2^2-\eta_1^2)/(\Phi N_B)\}^{1/2}$. ……(25)

Substitution of (25) in (19) gives $\bar{\sigma}=\sigma_W+C(1/N_B)^{1/2}$, ……(26)

where $\sigma_W=(2/3)H_1+(1/3)\sigma_p$. ……(27)

C is expressed by E_t as follows:

$$C=\{(2K_1/3)^2E_t/(\Phi\Psi)\}^{1/2},\ \Psi=\pi H_s(3c^2L^2+8\varepsilon_pcL+6\varepsilon_p^2)/\{4(2cL+3\varepsilon_p)\}. \cdots\cdots(28)$$

Adopting (21), this becomes $C=\{(2/3)^2K_1^2(\eta_2^2-\eta_1^2)/\Phi\}^{1/2}$, ……(29)

and $\Delta W=Kc\ (\eta_2^2-\eta_1^2)=\text{const.}$ ……(30)

Which is the condition of fracture from fatigue. cf. (65).

Now if the signs of the strains on one side are the same, we can have

$$0\leqq\sigma_p<H_1, \cdots\cdots(31)$$

so it must bring about the inequality $2H_1/3\leqq\sigma_W<H_1$, ……(32)

for the magnitudes of fatigue strengths from (27).

II. Model B

Contrary to the case of Model A, we consider a model in which the strains in the interior are larger than those on the periphery and in which the strains vary along an oblique straight curve against radial positions. This model will be called "Model B" in this paper. We keep the course and regard the max. strains and the distribution of them as well as the max. total stress as constant. Thus the value of strain ε_p on the periphery is constant, but the value of stress σ_p corresponding to the strain is variable with fatigue. Another treatment on the supposition that the value of the stress on the periphery is constant results in similar conclusions. By the aid of **Fig.2**, we have

$\varepsilon=\varepsilon_p+c(L-r),\ \varepsilon_p=\text{const.},\ c=\text{const.},$ ......(33)

$d\varepsilon=-cdr,\ c=\text{const.},$ ......(34)

$\text{OA}:\sigma=K(L-r)+\sigma_p,$ ......(35)

An: $H=Ka+\sigma_p$, where σ_p is variable with fatigue. ......(36)

Within the range $0\leqq\varepsilon\leqq\varepsilon_p$, we have

$\sigma=(K/c)\varepsilon$, where K and c are variables and K/c is constant. ......(37)

From (36) we have (8). Now we obtain from the expression of E similar to (10)

$$\Delta E=\{\pi cKL^4/(12H)+\pi\varepsilon_pKL^3/(3H)+\pi\varepsilon_p^2KL^2/(2cH)+\pi c\eta^3/3-\pi cKL\eta^3/(3H)-\pi\varepsilon_pK\eta^3/(3H)+\pi cK\eta^4/(4H)\}\Delta H+\{\pi cK^2L^4/(12H)+\pi\varepsilon_pK^2L^2/(3H)+\pi\varepsilon_p^2K^2L^2/(2cH)+\pi cH\eta^2-\pi cKL\eta^2-\pi\varepsilon_pK\eta^2+\pi cK\eta^3-\pi cK^2L\eta^3/(3H)-\pi\varepsilon_pK^2\eta^3/(3H)+\pi cK^2\eta^4/(4H)\}\Delta\eta. \quad\text{......(38)}$$

We have for the total stress

$$P=\pi KL^3+\pi(K/c)\varepsilon_pL^2+\pi H\eta^2-\pi KL\eta^2-\pi(K/c)\varepsilon_p\eta^2-2\pi KL^3/3+2\pi K\eta^3/3. \quad\text{......(39)}$$

and $\Delta P=0$ ......(40)

which is the condition of the total stress maintained constant which is reduced to

$\Delta H=(-K+H\varphi\eta^2)\Delta\eta,\ \varphi=6c/\{L^2(cL+3\varepsilon_p)\},$ ......(41)

where the terms of higher orders are neglected. Applying (41) to (38), we have

$\Delta E=\psi\eta^2\Delta\eta,\ \psi=\pi K(c^2L^2+4cL\varepsilon_p+6\varepsilon_p^2)/\{2(cL+3\varepsilon_p)\}.$ ......(42)

Arranging (39) by (16), we have $P=\pi L^2(H_1-2K_1L/3+K_1\eta_1),$ ......(43)

where $H=H_1,\ K=K_1$ at $\eta=\eta_1$, and $P=\pi L^2\bar{\sigma}.$ ......(44)

From (43) and (44) we have $\bar{\sigma}=H_1-2K_1L/3+K_1\eta,$ ......(45)

and we have from (42)

$E_u=\pi K(c^2L^2+4cL\varepsilon_p+6\varepsilon_p^2)\eta^3/\{6(cL+3\varepsilon_p)\},$ ......(46)

$E_t=\pi K(c^2L^2+4cL\varepsilon_p+6\varepsilon_p)(\eta_2^3-\eta_1^3)/\{6(cL+3\varepsilon_p)\}.$ ......(47)

Applying (46)(47)(22) and (24) to (45), we have $\bar{\sigma}=\sigma_W+C'(1/N_B)^{1/3},$ ......(48)

$\sigma_W=H_1-(2/3)K_1L,\ H_1/3\gtrsim\sigma_W<H_1,$ ......(49)

$C'=\{K_1^3 3E_t/(\Phi\Psi)\}^{1/3},\ \Psi=\pi K(c^2L^2+4cL\varepsilon_p+6\varepsilon_p^2)/\{2(cL+3\varepsilon_p)\}$ ......(50)

Now, adopting the expression of (47) as the form of E_t, and rewriting it by ΔW, we have $\bar{\sigma}=\sigma_W+C(1/N_B)^{1/2},$ ......(51)

$C=\{3/2)E\Delta W/\Phi\}^{1/2},\ \Delta W=Kc(\eta_2^2-\eta_1^2),$ E: Young's modulus. cf. (69) ......(52)

III. Discussion of the Condition of Fracture.

The signs 1 and 2 denote respectively the value before fatigue and just before fracture or vice versa. In the case of Model A we have from (13)

$$H_2=H_1[1+(\eta_1-\eta_2)/(L+\xi)+3\eta_2^2/\{L(2L+3\xi)\}-(L^2+3L\xi+3\xi^2)\eta_1^2/\{L(L+\xi)^2(2L+3\xi)-\eta_1\eta_2/(L+\xi)^2],\quad\text{......(53)}$$

where $\xi=\varepsilon_p/c=\text{const.}$

The signs 1 and 2 can be exchanged. The densities of strain energies at the points A_1, A_2, ... are as follows:

$W_{A1}=cH_1(L+\xi)/2-cH_1\eta_1/2,$ ......(54)

$W_{A2}=cH_2(L+\xi)/2-cH_2\eta_2/2,$ ......(55)

$$W_{A2}=cH_1[(L+\xi)+\eta_1-2\eta_2+(5L^2+9L\xi+3\xi^2)\eta_2^2/\{L(L+\xi)(2L+3\xi)\}-(L^2+3L\xi+3\xi^2)\eta_1^2/\{L(L+\xi)(2L+3\xi)\}-2\eta_1\eta_2/(L+\xi)/2,\quad\text{......(56)}$$

$W_{D1}=cH_1(L+\xi)/2-3cH_1\eta_1/2+3cH_1\eta_1^2/\{2(L+\xi)\},$ ......(57)

$$W_{D2}=c\{1-\eta_1/(L+\xi)\}(H_2^2/H_1)\{1-\eta_1/(L+\xi)\}(L+\xi)/2-c\{1-\eta_1/(L+\xi)\}\{(H_2^2/H_1)\cdot\{1-\eta_1/L+\xi)\}\eta_1/2,\quad\text{......(58)}$$

$$W_{D2}=cH_1[(L+\xi)-\eta_1-2\eta_2-6(L+\xi)\eta_1^2/\{L(2L+3\xi)+2\eta_1\eta_2/(L+\xi)+(8L^2+15L\xi+6\xi^2)\eta_2^2/\{L(L+\xi)(2L+3\xi)]/2,\quad\text{......(59)}$$

$$W_{B1}=cH_1\{1-\eta_2/(L+\xi)\}^2(L+\xi)H_1/(2H_2)-cH_1\{1-\eta_2/(L+\xi)\}^2\eta_2H_1/(2H_2),\quad\text{......(60)}$$

$$W_{B1}=cH_1[(L+\xi)-\eta_1-2\eta_2+3(L+\xi)\eta_1^2/\{L(2L+3\xi)\}+2\eta_1\eta_2/(L+\xi)-(L^2+3L\xi+3\xi^2)\eta_2^2/\{L(L+\xi)(2L+3\xi)]/2,\quad\text{......(61)}$$

$W_{B2}=cH_2\{1-\eta_2/(L+\xi)\}^2(L+\xi-\eta_2)/2,$ ......(62)

$$W_{B2}=cH_1[(L+\xi)+\eta_1-4\eta_2+3(5L^2+8L\xi+\xi^2)\eta_2^2/\{L(2L+3\xi)(L+\xi)-(L^2+3L\xi+3\xi^2)\eta_1^2/\{L(2L+\xi)(L+\xi)-4\eta_1\eta_2/(L+\xi)]/2.\quad\text{......(63)}$$

Applying (53) to (55) (58) (60) and (62), we get (56) (59) (61) and (63), and from the definition $\overline{\Delta W} = (W_{A_2} - W_{A_1}) + \{(W_{D_1} + W_{D_2}) - (W_{B_1} + W_{B_2})\}/2$, ... (64)

we have $\overline{\Delta W} = (L^2+3L\xi+3\xi^2)K_1c(\eta_2^2-\eta_1^2)/\{L(2L+3\xi)\}$. ....(65)

This is the same formula as (30) except a constant coefficient. The constant C of (29) can be written

$C = (hE\overline{\Delta W}/\Phi)^{1/2}$,

$h = 4L(2L+3\xi)/\{9(L^2+3L\xi+3\xi^2)\}$, E: Young's modulus. ....(66)

Now we calculate $\overline{\Delta W}$ in the case of Model B. On account of space consideration, we are giving the chief results only. In this case the condition of the total stress maintained constant is as follows:

$$H_2 = H_1[1+(\eta_1-\eta_2)/(L+\xi)+\eta_1^2/(L+\xi)^2-\eta_1\eta_2/(L+\xi)^2-(3L+\xi)\xi^2\eta_1^3/\{(L+\xi)^3L^2(L+3\xi)\}-\eta_1^2\eta_2/(L+\xi)^3+\eta_2^3/\{L^2(L+3\xi)\}]. \quad (67)$$

From (54)(55)(57)58)(60)(62)(64) and (67) we get[(3)]

$\overline{\Delta W} = K_1c(\eta_1^2-\eta_2^2)$, ...(68)

where $H_1/(L+\xi) \fallingdotseq K_1$. ....(69)

The right side of (68) coincides with that of ΔW of (52) except its sign. The same definition of $\overline{\Delta W}$ as in Part I and Part II can hold in Model A and Model B of this report too.

IV. Conclusion

In our idealized model, at least a bit of irregularity of stresses or the co-existence of plastic and elastic parts within the material is necessary for the occurrence of fatigue phenomena.

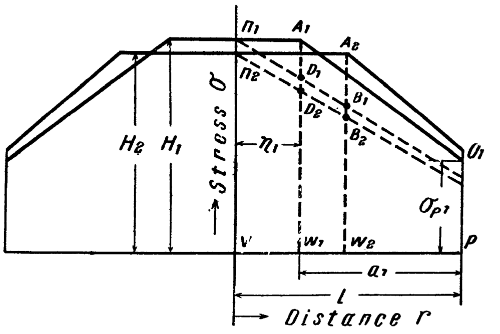

Fig. 2 Model B.

My theories hold good almost perfectly even in the case of the variation of η in the opposite direction to those reported in my papers which have been published previously, and it will depend upon further experiments towards, which direction the variation is found actually to occur in each case. The Discussions in Part I and Part II as well as in Part III were made chiefly on test-pieces subjected to completely reversed cycles of stress.

(3) When we use (57), we must add to it the terms of η^3 which has been neglected.