Post-Buckling Stability Analysis of Composite Laminated Flat Plates with Initial Deflection under Biaxial Compression*1

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Advanced fiber-reinforced laminated plates have been used as structural members in various applications by virtue of their high specific strength and stiffness. This paper considers, by use of the second variation of the total potential energy, secondary buckling of cross-ply laminated plates with an initial deflection under biaxial compression that is simply supported along four edges. The occurrence of secondary buckling is proven analytically and the effects of the initial deflection, outer lamination angle, number of layers, biaxial compressive load ratio and post-buckling deflection pattern are discussed. [doi:10.2320/matertrans.M2013111]

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1. Introduction

The use of advanced composite materials such as carbon-fiber-reinforced plastic has found an increasingly wide range of industrial applications because of their excellent properties such as a high specific strength and specific stiffness. In particular, the weight percentage of composite materials relative to the entire weight has significantly increased in civil aviation aircraft.

Although the buckling stress is one criterion for designing thin-walled structures, buckled structures often retain their load-carrying capacity without a decrease in the applied compressive load.1) Numerous researchers have discussed the post-buckling behavior of thin plates under in-plane compression.2,3) However, few researches have been conducted on the secondary buckling phenomenon, which occurs with a further increase in the load.4)

We previously proposed the secondary buckling behavior of cross-ply laminated plates5–7) and presented a post-buckling analysis of these plates with initial deflections,8) but the post-buckling stability under biaxial compressive loads is still unknown.

In this study, we examined the buckling problems of cross-ply laminated plates with initial deflections subject to biaxial compressive loads and expanded on the analytical techniques used previously.5–8) The stability of the post-buckling equilibrium states can be discussed in terms of the second variation of the total potential energy. Here, the occurrence of secondary buckling is proven analytically, and the effects of various factors are discussed.

2. Basic Equation of Orthotropic Laminated Plates

2.1 Basic relational equation

Figure 1 shows the coordinate system and dimensions of the symmetrically cross-ply laminated plates with initial deflections under biaxial compressive loads of \( N_x \) and \( N_y \). The displacement components \( u, v \) and \( w \) are those in the \( x, y \) and \( z \) directions, respectively, in the middle surface and \( \mathbf{w}_0 \) denotes the initial deflection. The in-plane nonlinear strain components \( \varepsilon_x, \varepsilon_y \) and \( \gamma_{xy} \) and the change in the curvatures \( \kappa_x, \kappa_y \) and \( \kappa_{xy} \) can be written in terms of the displacement components as follows.

\[
\begin{align*}
\varepsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 + \frac{\partial w}{\partial x} \left( \frac{\partial w}{\partial x} \right), \\
\varepsilon_y &= \frac{\partial u}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 + \frac{\partial w}{\partial y} \left( \frac{\partial w}{\partial y} \right), \\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} \left( \frac{\partial w}{\partial x} \right) + \frac{\partial w}{\partial y} \left( \frac{\partial w}{\partial y} \right), \\
\end{align*}
\]

(1)

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\[ \kappa_x = -\frac{\partial^2 w}{\partial x^2}, \quad \kappa_y = -\frac{\partial^2 w}{\partial y^2}, \quad \kappa_{xy} = -2 \frac{\partial^2 w}{\partial x \partial y}. \]

The axial, circumferential and shear strains of the median surface are given in terms of the median surface stresses in the usual form of Hooke's law for thin plates:

\[
\begin{align*}
N_x &= \int_{-h/2}^{h/2} \sigma_x dz = \frac{E_h}{1 - \nu_x \nu_y} \left[ \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial u}{\partial y} \right)^2 \right] + \nu_x \left( \frac{\partial u}{\partial y} \right) + \nu_y \left( \frac{\partial v}{\partial x} \right) + \frac{1}{2} \left( \frac{\partial v}{\partial y} \right)^2 \left( \frac{\partial v}{\partial y} \right) \\
N_y &= \int_{-h/2}^{h/2} \sigma_y dz = \frac{E_h}{1 - \nu_x \nu_y} \left[ \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial v}{\partial x} \right)^2 \right] + \nu_x \left( \frac{\partial v}{\partial x} \right) + \nu_y \left( \frac{\partial u}{\partial x} \right) + \frac{1}{2} \left( \frac{\partial u}{\partial x} \right)^2 \left( \frac{\partial u}{\partial x} \right) \\
N_{xy} &= \int_{-h/2}^{h/2} \tau_{xy} dz = G_{xy} \left[ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right] + \frac{1}{2} \left( \frac{\partial u}{\partial y} \right)^2 + \frac{1}{2} \left( \frac{\partial v}{\partial x} \right)^2 \left( \frac{\partial v}{\partial x} \right) \left( \frac{\partial u}{\partial y} \right) + \left( \frac{\partial u}{\partial x} \right) \left( \frac{\partial v}{\partial y} \right) \left( \frac{\partial v}{\partial x} \right). \end{align*}
\]

In this paper, the membrane force and bending moment resultants for the symmetrically cross-ply laminate are

\[
\begin{align*}
N_x &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \end{bmatrix}, \\
N_y &= \begin{bmatrix} 0 & 0 \\ 0 & A_{66} \end{bmatrix} \begin{bmatrix} \gamma_{xy} \end{bmatrix}, \\
M_x &= \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} \kappa_x \\ \kappa_y \end{bmatrix}, \\
M_y &= \begin{bmatrix} 0 & 0 \\ 0 & D_{66} \end{bmatrix} \begin{bmatrix} \gamma_{xy} \end{bmatrix}, \\
\text{where} \quad A_{ij} (i, j = 1, 2, 6) \text{ is the extensional stiffness, and } D_{ij} (i, j = 1, 2, 6) \text{ is the bending stiffness of the laminated plate; from these we can then calculate the material constants of the unidirectional composites } E_L, \ E_T, \ \nu_L, \ \nu_T, \ G_{LT} \text{ and the laminate angle } \theta. \text{ Suffixes L and T denote longitudinal and transverse, respectively, of the fiber.}
\end{align*}
\]

The energy \( U_m \) due to the strain in the middle of the plate and the energy \( U_b \) due to bending are given by

\[
\begin{align*}
U_m &= \frac{1}{2} \int_0^l \int_0^b \left( \frac{N_x}{E_x} \epsilon_x + \frac{N_y}{E_y} \epsilon_y + \frac{N_{xy}}{G_{xy}} \gamma_{xy} \right) dx dy, \\
U_b &= \frac{1}{2} \int_0^l \int_0^b \left( \frac{M_x}{\kappa_x} \kappa_x + \frac{M_y}{\kappa_y} \kappa_y + M_{xy} \gamma_{xy} \right) dx dy.
\end{align*}
\]

The total strain energy is obtained as a summation of these energies.

\[
\Pi = U_m + U_b.
\]

By substituting eqs. (3), (5), (6) and (7) into eq. (8), we obtain

\[
\Pi = \frac{1}{2} \int_0^l \int_0^b \left[ \frac{N_x^2}{E_x} + \frac{N_y^2}{E_y} + \frac{N_{xy}^2}{G_{xy}} \right] + \left[ \frac{D_{11} (\epsilon_x^2)}{\kappa_x^2} + 2 D_{12} (\epsilon_x^2 \epsilon_y^2) \right] + \left[ D_{22} (\epsilon_y^2) + 4 D_{66} (\epsilon_x^2 \epsilon_y^2)^2 \right] \right] dx dy.
\]

### 2.2 Equilibrium and compatibility equations

The equilibrium equations in the \( x \), \( y \) and \( z \) directions of a plate with an initial deflexion are expressed as follows:

\[
\begin{align*}
\epsilon_x &= \frac{\sigma_x}{E_x} - \frac{\nu_y \sigma_y}{E_y}, \quad \epsilon_y &= \frac{\sigma_y}{E_y} - \frac{\nu_x \sigma_x}{E_x}, \quad \gamma_{xy} = \frac{\tau_{xy}}{G_{xy}},
\end{align*}
\]

where \( \sigma_x \) and \( \sigma_y \) are the in-plane normal stresses and \( \tau_{xy} \) is the in-plane shear stress. \( E_x \) and \( E_y \) are the moduli of elasticity, \( \nu_x \) and \( \nu_y \) are Poisson’s ratios, and \( G_{xy} \) is the shear modulus. The membrane forces per unit width \( N_x, N_y \) and \( N_{xy} \) are defined as

\[
\begin{align*}
\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} &= 0, \\
\frac{\partial N_y}{\partial x} + \frac{\partial N_{xy}}{\partial y} &= 0, \\
D_{11} \frac{\partial^4 w}{\partial x^2 \partial y^2} + 2 (D_{12} + 2 D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^2} &= 0, \\
D_{11} \frac{\partial^4 w}{\partial x^2 \partial y^2} + 2 (D_{12} + 2 D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^2} &= 0.
\end{align*}
\]

The Airy function \( F \) that satisfies eq. (10) is defined by

\[
\begin{align*}
N_x &= h \frac{\partial^2 F}{\partial y^2}, \quad N_y &= h \frac{\partial^2 F}{\partial x^2}, \quad N_{xy} &= -h \frac{\partial^2 F}{\partial x \partial y},
\end{align*}
\]

and reduces the equilibrium in the lateral direction, eq. (11), to the form

\[
\begin{align*}
D_{11} \frac{\partial^4 w}{\partial x^2 \partial y^2} + 2 (D_{12} + 2 D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^2} &= 0, \\
D_{11} \frac{\partial^4 w}{\partial x^2 \partial y^2} + 2 (D_{12} + 2 D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^2} &= 0.
\end{align*}
\]

By eliminating \( u \) and \( v \) from the in-plane equilibrium of eq. (10), the following compatibility equation can be obtained:

\[
\begin{align*}
H_{12} = A_{22} A_{66} / H, \quad H_{22} = A_{12} A_{66} / H, \quad H_{66} = (A_{11} A_{22} - A_{12}^2) / H, \\
H = A_{44} A_{66} - A_{12} A_{22} = 0.
\end{align*}
\]
Post-buckling behavior can then be studied through the solutions of the nonlinear simultaneous equations of eqs. (9) and (10) involving \( F \) and \( w \) under the given boundary conditions. Approximate solutions of these equations will be sought because obtaining the exact solutions is very difficult.

2.3 Analysis of post-buckling behavior

The in-plane strain and membrane force are expressed as

\[
\begin{align*}
\sigma_{xx} &= \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{at} \quad x = 0, a, \\
\sigma_{yy} &= \frac{\partial^2 w}{\partial y^2} = 0 \quad \text{at} \quad y = 0, b, \end{align*}
\]

and the in-plane conditions are

\[
\begin{align*}
u &= \text{const. along } x, \\
N_x dx &= -N_y b, \quad N_{xy} = 0, \\
N_y dy &= -N_x a, \quad N_{xy} = 0. \end{align*}
\]

Here, \( w \) and \( w_0 \) are expressed by the following two terms that correspond to the symmetrical and asymmetrical modes, respectively, along the loading axis:

\[
\begin{align*}
w &= w_{11} \sin \left( \frac{\pi x}{a} \right) \sin \left( \frac{\pi y}{b} \right) + w_{21} \sin \left( \frac{2\pi x}{a} \right) \sin \left( \frac{\pi y}{b} \right), \\
w_0 &= c_{11} \sin \left( \frac{\pi x}{a} \right) \sin \left( \frac{\pi y}{b} \right) + c_{21} \sin \left( \frac{2\pi x}{a} \right) \sin \left( \frac{\pi y}{b} \right),
\end{align*}
\]

where \( w_{11} \) is the one half-wave number of the buckling half wave in both the \( x \)- and \( y \)-directions, and \( w_{21} \) is the two half-wave numbers of the buckling half wave in the \( x \) direction and the one half-wave number of the buckling half wave in the \( y \) direction. The \( c_{11} \) and \( c_{21} \) terms represent the amplitude of the initial deflection. Substituting eq. (16) into the right-hand side of eq. (14) yields

\[
F = \frac{1}{2h \lambda^2} \left( w_{11}^2 + 2w_{111c_11} \right)
\]

\[
\begin{align*}
&= \frac{\lambda^4}{16H_{12}^2} \cos \left( \frac{2\pi x}{a} \right) + \frac{1}{16H_{11}} \cos \left( \frac{2\pi y}{b} \right) \\
&+ \frac{2}{h \lambda^2} \left( w_{21}^2 + 2w_{21c_12} \right)
\end{align*}
\]

\[
\begin{align*}
&= \frac{\lambda^4}{256H_{12}} \cos \left( \frac{4\pi x}{a} \right) + \frac{1}{16H_{11}} \cos \left( \frac{2\pi y}{b} \right) \\
&- \frac{1}{4h \lambda^2} \left( w_{111w11} + w_{111c11} + w_{21c11} \right)
\end{align*}
\]

\[
\begin{align*}
&= \frac{\lambda^4}{H_{12}} \left[ \frac{1}{2} \cos \left( \frac{3\pi x}{a} \right) - \cos \left( \frac{\pi y}{b} \right) \right] \\
&+ \frac{\lambda^4}{H_{12}} \left( \frac{81H_{12} + 36(2H_{12} + H_{66}) \lambda^2 + 16H_{11} \lambda^4}{H_{12} + 4(2H_{12} + H_{66}) \lambda^2 + 16H_{11} \lambda^4} \right) - \frac{\sigma_{1y}^2 - \sigma_{2y}^2}{2},
\end{align*}
\]

where \( \lambda = a/b \).

When we substitute the deflection \( w \) and stress function \( F \) into the equilibrium equation in the lateral direction, i.e., eq. (13), the following simultaneous cubic equation is obtained by applying the Galerkin method. Both \( w_{11} \) and \( w_{21} \) are then determined by applying the Newton–Raphson method\(^8\) to eqs. (18) and (19). Substituting \( w \) and \( w_0 \) into eq. (11), we obtain the relationship between the average axial compressive stress \( \sigma_z \) and the average axial shortening \( \varepsilon_{\text{ext}} \) as follows:

\[
\varepsilon_{\text{ext}} = -\frac{1}{a} \int_0^a \left( \frac{\partial w}{\partial x} \right) dx
\]

\[
= -\frac{1}{a} \int_0^a \left( \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 - \frac{\partial w}{\partial x} \frac{\partial w_0}{\partial x} \right) dx
\]

\[
= H_{11} \sigma_z + \frac{\pi^2}{8a^2} (w_{111}^2 + 2w_{111c_11} + 4w_{21}^2 + 8w_{21c_12}).
\]

2.4 Stability criterion of post-buckling

The equilibrium equations can be obtained by equating the first variation \( \delta \Pi \) of the total potential energy \( \Pi \) to zero, i.e., the stationary potential energy criterion. The stability of the equilibrium state can be determined by the second variation \( \delta^2 \Pi \):

\[
\delta^2 \Pi = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \frac{1}{E_x} \left( \frac{\partial N_x}{\partial x} \right)^2 + \frac{1}{E_y} \left( \frac{\partial N_y}{\partial y} \right)^2 \\
- \left( \frac{\sigma_{1y} + \sigma_{1x}}{E_x} + \frac{\sigma_{2y} + \sigma_{2x}}{E_y} \right) \delta N_x \delta N_y + \frac{\sigma_{1y}^2 + \sigma_{2y}^2}{G_{xy}} \right]
\]

\[
+ \left( \frac{\partial w}{\partial y} \frac{\partial w}{\partial y} \right) \left( \frac{\partial w}{\partial x} \frac{\partial w}{\partial x} \right) + \frac{\partial w_0}{\partial y} \frac{\partial w_0}{\partial y} \right] \int_0^a \left( \frac{\partial w}{\partial x} \frac{\partial w}{\partial x} \right) dx dy,
\]

where

\[
\delta N_x = \frac{E_x h}{1 - \nu_1 \nu_2} \left[ \frac{\partial w}{\partial x} \left( \frac{\partial w}{\partial x} \right) + \frac{\partial w_0}{\partial x} \left( \frac{\partial w_0}{\partial x} \right) \right] \]

\[
+ \frac{\partial w}{\partial y} \left( \frac{\partial w}{\partial y} \right) + \frac{\partial w_0}{\partial y} \left( \frac{\partial w_0}{\partial y} \right),
\]

\[
\delta N_y = \frac{E_y h}{1 - \nu_1 \nu_2} \left[ \frac{\partial w}{\partial y} \left( \frac{\partial w}{\partial y} \right) + \frac{\partial w_0}{\partial y} \left( \frac{\partial w_0}{\partial y} \right) \right] \]

\[
+ \frac{\partial w}{\partial x} \left( \frac{\partial w}{\partial x} \right) + \frac{\partial w_0}{\partial x} \left( \frac{\partial w_0}{\partial x} \right),
\]

\[
\delta N_{xy} = G_{xy} h \left[ \frac{\partial w}{\partial x} \frac{\partial w}{\partial x} + \frac{\partial w_0}{\partial x} \frac{\partial w_0}{\partial x} \right] + \left( \frac{\partial w}{\partial y} \right) \left( \frac{\partial w_0}{\partial y} \right) \]

\[
+ \left( \frac{\partial w}{\partial x} \right) \left( \frac{\partial w_0}{\partial x} \right) + \left( \frac{\partial w}{\partial y} \right) \left( \frac{\partial w_0}{\partial y} \right).
\]

If \( \delta^2 \Pi \) is positive, the equilibrium state is stable; if it is zero, the equilibrium state is neutral; and if it is negative, the equilibrium state is unstable. An assumed virtual displacement poses a problem because the sign of \( \delta^2 \Pi \) cannot be
judged independently of the assumed patterns. In this study,
the proposed method was used to set the appropriate virtual
displacements. That is, the variation of $\delta^2\Pi$ with respect to
$\delta u$, $\delta v$ and $\delta w$ is considered essential for obtaining
the extreme value of $\delta^2\Pi$ under the given boundary conditions
for a set of assumed virtual displacements. From $\delta^2\Pi$, the
following equilibrium equations for the virtual displacements
are obtained:

$$\begin{align*}
\frac{\partial\delta N_x}{\partial x} + \frac{\partial\delta N_y}{\partial y} &= 0, \\
\frac{\partial\delta N_y}{\partial x} + \frac{\partial\delta N_z}{\partial y} &= 0,
\end{align*}$$

(23)

$$D_{11} \frac{\partial^2 \delta w}{\partial x^2} + 2\left(\frac{D_{12} + D_{66}}{2}\right) \frac{\partial^2 \delta w}{\partial x \partial y} + D_{22} \frac{\partial^2 \delta w}{\partial y^2} = \delta N_x \frac{\partial^2 (w + w_0)}{\partial x^2} + 2\delta N_y \frac{\partial^2 (w + w_0)}{\partial x \partial y} + \delta N_y \frac{\partial^2 (w + w_0)}{\partial y^2} + N_x \frac{\partial^2 \delta w}{\partial x^2} + 2N_y \frac{\partial^2 \delta w}{\partial x \partial y} + N_y \frac{\partial^2 \delta w}{\partial y^2}.
$$

(24)

By introducing the virtual stress function $\delta F$ that satisfies
eq (23),

$$\delta N_x = \frac{h}{4\pi} \frac{\partial^2 \delta F}{\partial x^2}, \quad \delta N_y = \frac{h}{4\pi} \frac{\partial^2 \delta F}{\partial x \partial y}, \quad \delta N_y = -\frac{h}{4\pi} \frac{\partial^2 \delta F}{\partial y^2},
$$

(25)

and by eliminating $\delta u$ and $\delta v$ from eq. (22), the following
compatibility equation for the virtual displacements is obtained:

$$H_{22} \frac{\partial^4 \delta F}{\partial x^4} + (2H_{12} + H_{66}) \frac{\partial^2 \delta F}{\partial x^2 \partial y^2} + H_{11} \frac{\partial^2 \delta F}{\partial y^4} = \frac{1}{h} \left\{ 2 \left( \frac{\partial^2 \delta w}{\partial x^2} \right) \left( \frac{\partial^2 \delta w}{\partial y^2} \right) - \frac{\partial^2 \delta w}{\partial x^2} \frac{\partial^2 \delta w}{\partial y^2} + \frac{\partial^2 \delta w}{\partial x^2} \frac{\partial^2 \delta w}{\partial y^2} \right\}.
$$

(26)

For $\delta^2\Pi = 0$, the boundary conditions of the infinitesimal
virtual displacements $\delta u$, $\delta v$ and $\delta w$ and the virtual
membrane forces $\delta N_x$, $\delta N_y$, $\delta N_y$ are

$$\delta u = 0, \quad \delta N_x = 0, \quad \delta N_y = \frac{\partial^2 \delta w}{\partial y^2} = 0 \text{ at } x = 0, a,
$$

(27)

$$\delta v = 0, \quad \delta N_y = 0, \quad \delta u = \frac{\partial^2 \delta w}{\partial x^2} = 0 \text{ at } y = 0, b,
$$

and for the simply supported condition, $\delta w$ is assumed to be

$$\delta w = \delta w_{pq} \sin \frac{a \pi x}{p} \sin \frac{b \pi y}{q},
$$

(28)

where $p$ and $q$ are positive integers. This corresponds to the
predominant deflected mode of the post-buckled state, which
is excluded because it does not satisfy the $\delta u$ and $\delta v$
conditions in eq. (27). Substituting $w$, $w_0$ and $\delta w$ of eqs. (18)
and (28) into eq. (26), $\delta F$ is determined so as to satisfy
eq (27).
where $N_x$, $N_y$ and $N_{xy}$ are determined by eqs. (12) and (19), and the virtual membrane forces $\delta N_x$, $\delta N_y$ and $\delta N_{xy}$ are determined by eqs. (25) and (29). Furthermore, substituting $w$ and $w_0$ of eq. (18) and $\delta w$ of eq. (28) into eq. (21) yields the final equation for $\delta^2 \Pi$.

3. Numerical Results and Discussion

We applied this solution method to investigate the secondary buckling behavior of cross-ply laminated plates with initial deflections. The elastic constants of unidirectional carbon fiber-reinforced epoxy composites having a fiber volume content of $V_f = 60\%$, obtained using the approximate average method, were confirmed by experiments\(^\text{10}\) and are as follows:

\[
\begin{align*}
E_L &= 137 \text{ (GPa)}, \\
E_T &= 8.17 \text{ (GPa)}, \\
G_{LT} &= 4.75 \text{ (GPa)}, \\
\nu_L &= 0.316, \\
\nu_T &= 0.0189.
\end{align*}
\]

(30)

The relationship between the non-dimensional average axial compression $K$ ($= \sigma x h^2/E T h^2$) and the average axial strain in the $x$-direction $\varepsilon_{mx}$ of the orthotropic cross-ply laminated square plates ($\lambda = 1$) without initial deflections is shown in Fig. 2. The $k_x = 0$ (i.e., uniaxial compression) case is shown by the solid line, the $k_x = 0.5$ case by the dash-dot line, and the $k_x = 1.0$ case by the dash-dot-dot line. In this figure, $K_p$ is the primary buckling value, and $K_s$ is the secondary buckling value. The number of primary buckling half waves and secondary buckling two half waves in the $x$- and $y$-directions are indicated in parentheses.

The relationship between the non-dimensional average axial compression $K$ and average axial strain of the $x$-axis direction $\varepsilon_{mx}$ is shown in Figs. 3 to 5.

Figure 3 shows the relationship between $K$ and $\varepsilon_{mx}$ for $\lambda = 1.0$ when $k_y = 0$ ($= N_{xy}/N_y = 0.0$, i.e., a biaxial compressive load ratio corresponding to uniaxial compression. Figures 4 and 5 show the same relationship for biaxial compressive load ratios $k_y$ of 0.5 and 1.0, respectively.

The $c_{11}$ and $c_{21}$ values are the initial deflection amplitudes of the symmetrical and asymmetrical modes based on eq. (18). Furthermore, $c_{11} = 0.1$ is the one half wave in the $x$-direction and the one half wave in the $y$-direction and indicates that the magnitude is 10% of the plate thickness.

Table 1 lists the variation of the secondary buckling stresses for orthotropic cross-ply laminated plates with initial deflections of $c_{11} = 0$, $0.05$ and $0.10$ under a biaxial compressive load ratio of $k_y = 1.0$. The analytical method
indicated that the second variation of the total potential energy $\delta^2\Pi$ depends strongly on the number of half waves $p$ and $q$ in the infinitesimal virtual displacement; the minimum values are then the secondary buckling stress.

Tables 2–4 list the secondary buckling stresses and number of half waves of infinitesimal virtual displacements $p$ and $q$ for outer lamination angles of $\theta = 0$ and $90^\circ$ and 3 or 5 layers, as well as those for an infinite number of layers (orthotropic lamination), for initial deflections of $c_{11} = 0$, 0.05 and 0.10 and $c_{21} = 0$, 0.05 and 0.10. Table 2 lists the results for $k_y = 0.0$ and Tables 3 and 4 are those for $k_y = 0.5$ and 1.0, respectively.

As shown in Table 2, for an outer lamination angle of $\theta = 0^\circ$ and 3 layers, $w_{11}$ governs the conditions regarding the deflection pattern of the post-buckling. The one half-wave number of the primary buckling in the $x$- and $y$-direction are determined. Thus, in this case, $w_{21}$ is ignored. In contrast, for an outer lamination angle of $\theta = 90^\circ$ and 3 layers, $w_{21}$ governs the conditions. The two half-wave number of primary buckling in the $x$-direction and one half-wave number of primary buckling in the $y$-direction are determined. Thus, in this case $w_{11}$ is ignored.

As seen in the tables, minimum values of $K_s$ are obtained when infinitesimal virtual displacements are considered regardless of the amplitude of the initial deflection.
The post-buckled equilibrium state with small-wave modes $p$ and $q$ (in both the $x$- and $y$-directions) cannot be proved to be unstable by using an infinitesimal virtual displacement.

It can be proved that the post-buckled equilibrium state becomes unstable when almost infinitesimal virtual displacements $p$ and $q$ are considered and the inevitable secondary buckling is given analytically.

As shown in Figs. 2–5 and Tables 2–4, the secondary buckling stress was several times that of the primary buckling stress; the stresses were obtained for the load-carrying capacity of thin laminated plates after primary buckling. The curves in Figs. 2–5 become straight lines with a slope equal to that prior to buckling. After primary buckling, the slope of this line may decrease. This indicates a loss of stiffness in the post-buckling range and is greatly dependent on the number of half-wave of the infinitesimal virtual displacements $p$ and $q$.5)

It is seen from Tables 2–4 that the secondary buckling $K_s$ increases as the number of layers increases. There is a difference in the increasing rates of the secondary buckling as a function of the number of layers between the outer lamina angles of 0 and 90°. The secondary buckling values for outer lamina angles of 0 and 90° agree for an infinite number of layers.

The secondary buckling also decreases with increasing biaxial compressive load ratios, and there was little change in the secondary buckling stress within the ranges of initial deflection amplitudes considered here. In addition, the secondary buckling stresses with an initial deflection were nearly the same as those without any initial deflection, and the secondary buckling $K_s$ increased only slightly as the amplitude of the initial deflection increased. Accordingly, the initial deflection was 10% of the plate thickness; the effect on the secondary buckling stresses was negligible.

4. Conclusions

We have shown analytically that the post-buckling equilibrium states for simply supported cross-ply laminated plates with initial deflections under biaxial compressive loads can be unstable. We proposed a method based on the secondary variation of the total potential energy to evaluate the stability of the post-buckling equilibrium states, and the secondary buckling was derived. The effect of initial deflections and biaxial compressive load ratios on the post-buckling equilibrium states and secondary buckling stresses were discussed.

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REFERENCES