Postbuckling Behavior of Composite Laminated Plates with Initial Imperfections under Biaxial Compression

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Advanced fiber-reinforced laminated plates have been used for structural members in various fields, by virtue of their high specific strength and stiffness. This paper considers, by use of Galerkin’s methods, the postbuckling behaviors of angle-ply laminated plates with initial deflection under biaxial compression that is simply supported along four edges. The inevitability of postbuckling behaviors is proved analytically, and the effects of various factors, such as initial imperfection, lamination angle, biaxial compressive load ratio, and postbuckling deflection pattern, are clarified. [doi:10.2320/matertrans.MRA2008250]

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1. Introduction

In recent years, laminated plates have become important structural elements in various engineering applications. Especially in aerospace applications where savings in weight are of great importance, advanced fiber-reinforced composite materials are widely used for laminated fiber-reinforced plates and other structural shapes. The composite materials have high in-plane strength but low density. Hence, high strength and high stiffness-to-weight ratios are readily obtained. As is well-known, postbuckling thin plates are stable and hence remain structurally useful well beyond their primary Euler buckling loads. Postbuckling behaviors of thin laminated plates under uniaxial compression have been discussed by numerous researchers. However, little research has been performed on the secondary buckling phenomenon for thin laminated plates, which occurs with further increases in load.

This paper addresses postbuckling problems of orthotropic angle-ply laminated plates with small initial deflection subjected to in-plane biaxial compression, which are simply supported along four edges. A theoretical analysis is used to predict the inevitable postbuckling behavior of angle-ply laminated plates by deflected patterns and initial displacement patterns. Numerical solutions are obtained for average axial strain, biaxial compression ratio of the square plate, and initial imperfections, which are illustrated graphically and discussed.

2. Fundamental Equations

2.1 Basic equation

The coordinate system and dimensions of a square plate under biaxial compression are shown in Fig. 1. \( u, v \) and \( w \) denote the displacement components in \( x, y \) and \( z \) directions, respectively, at the middle surface, and \( w_0 \) denotes the initial deflection. The in-plane nonlinear strain components \( \varepsilon_x, \varepsilon_y \) and \( \gamma_{xy} \) and the changes of curvatures \( \kappa_x, \kappa_y \) and \( \kappa_{xy} \) can be written in terms of displacement components as follows.

\[
\begin{align*}
\varepsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 + \frac{\partial w_0}{\partial x}, \\
\varepsilon_y &= \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 + \frac{\partial w_0}{\partial y}, \\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} + \frac{\partial w}{\partial y} \frac{\partial w}{\partial x} + \frac{\partial w_0}{\partial y} \frac{\partial w_0}{\partial x}, \\
\kappa_x &= -\frac{\partial^2 w}{\partial x^2}, \\
\kappa_y &= -\frac{\partial^2 w}{\partial y^2}, \\
\kappa_{xy} &= -2 \frac{\partial^2 w}{\partial x \partial y}.
\end{align*}
\]

The axial, circumferential, and shear strains in the median surface are given in terms of the median surface stresses by the usual form of Hooke’s law for thin plates.

\[
\begin{align*}
\varepsilon_x &= \frac{\sigma_x}{E_x} - \nu_y \frac{\sigma_y}{E_y}, \\
\varepsilon_y &= \frac{\sigma_y}{E_y} - \nu_x \frac{\sigma_x}{E_x}, \\
\gamma_{xy} &= \frac{\tau_{xy}}{G_{xy}},
\end{align*}
\]

where \( E_x \) and \( E_y \) are moduli of elasticity, \( \nu_x \) and \( \nu_y \) are Poisson’s ratios, and \( G_{xy} \) is the shear modulus. The membrane forces per unit width \( N_x, N_y \) and \( N_{xy} \) are defined as follows:

Fig. 1 Configuration and coordinates of angle-ply laminated plate with small initial deflection under biaxial compressive loads.
\[ N_x = \int_{-h/2}^{+h/2} \sigma_x dz, \quad N_y = \int_{-h/2}^{+h/2} \sigma_y dz, \quad N_{xy} = \int_{-h/2}^{+h/2} \tau_{xy} dz. \]  
(4)

The membrane force and bending moment for an orthotropic angle-ply laminate are

\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy}
\end{bmatrix} = \begin{bmatrix}
A_{11} & A_{12} & 0 \\
A_{12} & A_{22} & 0 \\
0 & 0 & A_{66}
\end{bmatrix} \begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix},
\end{align}
\]
(5)

where \(A_{ij}\) (\(i, j = 1, 2, 6\)) are the extensional stiffness and \(D_{ij}\) \((i, j = 1, 2, 6)\) are the bending stiffness of laminated plate, from which we can calculate material constants of unidirectional composites \(E_l, E_T, v_l, v_T, G_{LT}\) and lamination angle \(\theta\) deg.

### 2.2 Equilibrium and compatibility equations

The equilibrium equations in the \(x\), \(y\) and \(z\) directions of a plate with an initial imperfection are expressed as follows:

\[
\begin{aligned}
\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} &= 0, \\
\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} &= 0,
\end{aligned}
\]
(6)

\[
\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + N_x \frac{\partial^2 (w + w_0)}{\partial x^2} \\
+ 2N_{xy} \frac{\partial^2 (w + w_0)}{\partial x \partial y} + N_y \frac{\partial^2 (w + w_0)}{\partial y^2} &= 0.
\]
(7)

The Airy function \(F\), which satisfies eq. (6), is defined by

\[ N_x = h \frac{\partial^2 F}{\partial y^2}, \quad N_y = h \frac{\partial^2 F}{\partial x^2}, \quad N_{xy} = -h \frac{\partial^2 F}{\partial x \partial y}, \]
(8)

and reduces the equilibrium equation to the lateral direction to the form of

\[
\frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 w}{\partial y^4} \\
= \frac{h}{C_{18}/C_{19}} \left\{ \frac{\partial^2 F}{\partial x^2} \frac{\partial^2 (w + w_0)}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} \frac{\partial^2 (w + w_0)}{\partial y^2} \right\} \\
- \frac{2}{C_{0} \partial x \partial y} \frac{\partial^2 (w + w_0)}{\partial x \partial y}.
\]
(9)

The compatibility equation is given as

\[
\frac{\partial^2 \varepsilon_x}{\partial y^2} - \frac{\partial^2 \varepsilon_y}{\partial x \partial y} + \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \\
= \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \\
+ 2 \frac{\partial^2 w}{\partial x \partial y} \frac{\partial w_0}{\partial x \partial y} - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w_0}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \frac{\partial^2 w_0}{\partial y^2}.
\]
(10)

In case of homogeneous, isotropic plate without initial deflection, the equations reduce to the well known ones originally derived by von Kármán,

\[
\nabla^4 F = E \left[ \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right].
\]
(9a)

The average axial strain of \(x\)-axis direction \(\varepsilon_{mx}\) in the middle of the surface of the plate is defined by

\[
\varepsilon_{mx} = -\frac{1}{a} \int_0^a \left( \frac{\partial w}{\partial x} \right)^2 dx,
\]
(11)

from which we can obtain the relation between the non-dimensional average compression \(K = (\sigma_c b^2 / E h^2)\), where \(\sigma_c = N_c / h\) and the average axial strain of \(x\)-axis direction \(\varepsilon_{mx}\).

Postbuckling behavior can then be studied through the solution of non-linear simultaneous eqs. (9) and (10) involving \(F\) and \(w\) under the given boundary conditions. Approximate solutions of these equations will be sought, since obtaining the exact solutions is very difficult.

### 2.3 Analysis of postbuckling behaviors

The in-plane strain and the membrane force are expressed as follows from eq. (4),

\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix} = \begin{bmatrix}
H_{11} & H_{12} & 0 \\
H_{12} & H_{22} & 0 \\
0 & 0 & H_{66}
\end{bmatrix} \begin{bmatrix}
N_x \\
N_y \\
N_{xy}
\end{bmatrix},
\]
(12)

where compliance stiffness \(H_{ij}\) \((i, j = 1, 2, 6)\) is expressed as follows:

\[
H_{11} = A_{22} A_{66}/H, \quad H_{12} = -A_{12} A_{66}/H, \quad H_{22} = A_{11} A_{66}/H, \\
H_{66} = (A_{11} A_{22} - A_{12}^2)/H, \quad H = A_{11} A_{22} A_{66} - A_{12} A_{66}^2.
\]
(13)

The simply supported out-plane boundary conditions are

\[
w = \frac{\partial^2 w}{\partial y^2} = 0 \text{ at } x = 0, a, \]
(14)

\[
w = \frac{\partial^3 w}{\partial x \partial y^2} = 0 \text{ at } y = 0, b,
\]
and the in-plane boundary conditions are

\[
a = \text{constant, along the } y\text{-axis, } \left( \int_0^a N_x dy = -P_x \right), \quad \varepsilon_{xy} = 0 \text{ at } x = 0, a, \]
(15)

\[
v = \text{constant, along the } x\text{-axis, } \left( \int_0^a N_x dx = -P_y \right), \quad \varepsilon_{xy} = 0 \text{ at } y = 0, b.
\]

Here, \(w\) and \(w_0\) are expressed by the following two terms that correspond to symmetrical and asymmetrical modes along the loading axis, respectively

\[
\begin{bmatrix}
\frac{\pi x}{a} \sin \frac{\pi y}{b} + w_1 \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \\
\frac{2\pi x}{a} \sin \frac{\pi y}{b} + w_2 \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}
\end{bmatrix}
\]
(16)

where, \(w_{11}\) means the one half wave number of buckling half wave in \(x\) direction and the one half wave number of buckling half wave in \(y\) direction, and \(w_{21}\) means the two half waves number of buckling half wave in \(x\) direction and the one half wave number of buckling half wave in \(y\) direction.
wave number of buckling half wave in \( y \) direction. Moreover, \( c_{11} \) and \( c_{21} \) represent deflection of initial imperfection. The introduction of eq. (16) into the right-hand side of eq. (10) yields:

\[
F = \frac{1}{2h^2} (w_{11})^2 + 2w_{11}c_{11}) \\
\times \left\{ \frac{A^4}{16H_{22}} \cos \left( \frac{2\pi x}{a} \right) + \frac{1}{16H_{11}} \cos \left( \frac{2\pi y}{b} \right) \right\} \\
+ \frac{2}{k^2} (w_{21})^2 + 2w_{21}c_{21}) \\
\times \left\{ \frac{A^4}{256H_{22}} \cos \left( \frac{4\pi x}{a} \right) + \frac{1}{16H_{11}} \cos \left( \frac{2\pi y}{b} \right) \right\} \\
- \frac{1}{4h^2} (w_{11}w_{21} + w_{11}c_{21} + w_{21}c_{11}) \\
\times \left\{ \frac{\lambda^4}{H_{22}} \left\{ \frac{1}{9} \cos \left( \frac{3\pi x}{a} \right) - \cos \left( \frac{\pi y}{b} \right) \right\} - \frac{\lambda^4 \cos(3\pi x/a) \cos(2\pi y/b)}{81H_{22} + 36(2H_{12} + H_{66})\lambda^2 + 16H_{11}\lambda^4} + \frac{\lambda^4 \cos(\pi x/a) \cos(2\pi y/b)}{H_{22} + 4(2H_{12} + H_{66})\lambda^2 + 16H_{11}\lambda^4} \right\} \\
- \frac{\sigma_{xy}^2}{2} - \frac{\sigma_{xy}^2}{2}, \quad (17)
\]

where \( \lambda = a/b \).

When we substitute deflection \( w \) and stress function \( F \) for the equilibrium equation in the lateral direction; i.e., as eq. (9), the following simultaneous cubic equations are obtained by applying the Galerkin method:

\[
\pi^2 \left( \frac{w_{11}}{a^2k_x} \right) \left\{ D_{11} + 2(D_{12} + 2D_{66})\lambda^2 + D_{22}\lambda^4 \right\} + \frac{\pi^4}{32} \left( \frac{1}{ab} \right) \\
\times \left\{ \frac{1}{H_{11}\lambda^2} (w_{11} + c_{11})((w_{11})^2 + 2w_{11}c_{11}) + 4(w_{21})^2 + 2w_{21}c_{21}) \right\} \\
+ \frac{1}{H_{22}} \lambda^2 (w_{11} + c_{11})(w_{11})^2 + 2w_{11}c_{11}) \\
\times \left\{ \frac{1}{H_{22}} + 4(2H_{12} + H_{66})\lambda^4 + 16H_{11}\lambda^4 \right\} \\
\times \left\{ \frac{8}{3} w_{11}(w_{11} + c_{11}) \right\} \\
\times \left( \frac{128}{5} w_{21}(w_{21} + c_{21}) \right) 16H_{22} + 4(2H_{12} + H_{66})\lambda^2 + 16H_{11}\lambda^4 \\
\times \left\{ \frac{1}{H_{22}} \left\{ \frac{1}{6} (w_{11})^2 + 2w_{11}c_{11}) + \frac{1}{30} (w_{21})^2 + 2w_{21}c_{21}) \right\} \right\} \\
- \frac{\sigma_{xy}^2}{4k_x} (w_{11} + c_{11}) (1 + \lambda^2k_y) = 0, \quad (18)
\]

\[
\pi^4 \left( \frac{w_{21}}{a^2k_x} \right) \left\{ 16D_{11} + 8(D_{12} + 2D_{66})\lambda^2 + D_{22}\lambda^4 \right\} + \frac{\pi^4}{64} \left( \frac{A^2}{b^2} \right) \\
\times \left\{ \frac{4}{H_{11}\lambda^2} (w_{21} + c_{21})((w_{21})^2 + 2w_{21}c_{21}) \right\} \\
+ \frac{1}{H_{22}} (w_{21} + c_{21})(w_{21})^2 + 2w_{21}c_{21}) \\
\times \left\{ \frac{1}{H_{22}} + 4(2H_{12} + H_{66})\lambda^4 + 16H_{11}\lambda^4 \right\} \\
\times \left\{ \frac{128}{5} w_{21}(w_{11} + c_{11}) \right\} \\
\times \left( \frac{32}{5} w_{11}(w_{21} + c_{21}) \right) \right\} \\
\times \left\{ \frac{1}{H_{22}} + 4(2H_{12} + H_{66})\lambda^2 + 16H_{11}\lambda^4 \right\} \\
\times \left\{ \frac{1}{2} (w_{21} + w_{21}c_{21}) \right\} \\
\times \left\{ \frac{1}{6} 81H_{22} + 36(2H_{12} + H_{66})\lambda^2 + 16H_{11}\lambda^4 + \frac{16}{15H_{22}} \right\} \\
- \frac{\sigma_{xy}^2}{\lambda} (w_{21} + c_{21}) \left\{ 1 + \lambda^2k_y \right\} = 0, \quad (19)
\]
where, $k_y (= \sigma_y / \sigma_z)$ is biaxial compressive load ratio.

Then $w_{11}$ and $w_{21}$ are determined by applying the Newton-Raphson method to eqs. (18) and (19). Substituting $w_0$ into eq. (11), we obtain the relationship between the average axial compressive stress $\sigma_{x}$ and the average axial shortening $\epsilon_{mx}$, as follows:

$$\epsilon_{mx} = \frac{\pi^2}{8a^2} (w_{11})^2 + 2w_{11}c_{11} + 4w_{21}^2 + 8w_{21}c_{21}).$$

(20)

3. Numerical Results and Discussion

The postbuckling behavior of a square angle-ply laminated plate with initial deflection, which is simply supported on four edges under biaxial compression, is analyzed by the procedure mentioned above. The effects of various factors on the postbuckling stresses are discussed herein.

The elastic constants of unidirectional carbon fiber-reinforced epoxy composites with fiber volume content $V_f = 60\%$ used for numerical example are as follows.

$E_L = 137$ (GPa), $E_T = 8.18$ (GPa), $G_{LT} = 4.75$ (GPa), $v_L = 0.316$, $v_T = 0.0189$.

These values can be obtained by explicit algebraic equations when the constituent materials and $V_f$ are specified, and they are confirmed by experiments. 10)

The method of solution outlined in the foregoing will be applied to investigate the postbuckling behavior of square angle-ply laminated plates.

Below, numerical solutions will be obtained for initially flat plates and plates with initial deflection, of, for example, $c_{11}$, the magnitude of which is defined as 0.05 or 0.10h.

The relationship between non-dimensional average axial compression $K (= \sigma_{x}b^2/E_T h^2)$ and average axial strain of $x$-axis direction $\epsilon_{mx}$ is shown in Figs. 2 to 4, where $k_y (= N_y/N_x)$ is biaxial compressive load ratio.
Figure 2 shows the relationship between $K$ and $\varepsilon_{\text{mx}}$ at $\lambda$ ($=a/b$) = 1.0, when $k_y = 0.0$; that is, the biaxial compressive ratios corresponding to the uniaxial compression. The cases of $\theta = 0$, 30, 45, 60, 90 deg. with initial deflection $c_{11} = 0.05$ and 0.10, $c_{21} = 0.05$ and 0.10 are shown in the form of non-dimensional average axial compression $K$ and average axial strain of $x$-axis direction $\varepsilon_{\text{mx}}$ in Figs. 2(a), (b), (c), (d) and (e), respectively.

Moreover, Figs. 3 and 4 show $K$ and $\varepsilon_{\text{mx}}$ at $\lambda$ ($=a/b$) = 1.0, when the biaxial compressive ratios $k_y$ are 0.5 and 1.0, respectively.

In all figures, non-imperfection $c_{11} = 0.0$ and $c_{21} = 0.0$ are illustrated by solid lines, $c_{11}$ or $c_{21} = 0.05$ is illustrated by a line of alternating dashes and dots, and $c_{11}$ or $c_{21} = 0.10$ is illustrated by chain double-dotted line.

As shown in Figs. 2 to 4, this curve becomes a straight line prior to primary buckling, independent of initial deflection. After buckling, the slope of this line may decrease, indicating a loss of stiffness in the postbuckling range and exhibiting great dependence upon the number of half-waves of $m$ and $n$. As for effect of lamination angle $\theta$ deg., the maximum slope of postbuckling is indicated at $\theta = 45$ deg., and the slope decreases with increasing lamination angle $\theta$ deg.

As for effect of biaxial compression ratio $k_y$, we can see from Figs. 2 to 4 that the postbuckling $K$ decreases with increasing biaxial compression ratio for a square plate.

One of authors indicated the postbuckling behaviors of angle-ply laminated plates without imperfections;\(^{11}\) as for effects of initial deflection, as shown in Figs. 2 to 4, the postbuckling $K$ decreased with increasing initial deflection. It was thought it was unquestionable for this analytical technique (including algorithm), and we evaluated the validity of the solution because it had agreed to the content shown in reference 12) since this analytical technique was applied to an isotropic plate with initial deflection.
4. Conclusion

We show analytically that the postbuckling behavior for simply supported angle-ply laminated plates with small initial curvature can be proved.

We considered square plates in this study. With regard to the effect of plate aspect ratio $a/b$, in future studies we will analyze postbuckling behavior by the theoretical method.

Authors extended an analytical technique\(^{12}\) of isotropic plates with initial imperfections to an-isotropic plates with initial imperfections, and postbuckling behavior of composite laminated plates with initial imperfections, are clarified.

The future direction of this study will be analysis of postbuckling behavior of angle-ply laminated rectangular plates with initial imperfection and secondary buckling of angle-ply laminated square plates with initial imperfection. Additionally, we would like to conduct optimal design for secondary buckling of composite laminated plates.

REFERENCES