Heat Transfer Model for Thin Solidified Material in Continuous Casting

Nobuaki Ito

Advanced Technology Research Laboratories, Nippon Steel Corporation, Fattsu 293-8511, Japan

An asymptotic explicit numerical method was developed for the Stefan problem in which a series of solidification rates and boundary temperatures for the solidified material are given and the boundary heat flux is returned. A spectral method with several basis functions of a specialized shape in the solidification problem was adopted. Combined with multi-dimensional computational fluid dynamics methods for the liquid zone, this method is adequate for resolving the thin solidified material problem for a variety of continuous casting processes e.g. thin slab continuous casting, melt spinning, twin roll casting, and edge-defined film-fed growth. The method is less expensive than conventional numerical methods and as accurate as a direct numerical approach such as the Finite Difference Method especially in the case of the Stefan number \( \gg 1 \) or in the case of variable material properties. [doi:10.2320/matertrans.MRA2007019]

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1. Introduction

Heat and flow analysis in continuous casting (C.C.) around the solidification front (S.F.) provides valuable information on appropriate working conditions and morphological control of casting. As shown previously, the multi-domain approach divided with discontinuous S.F. should be applied in the C.C. simulation from the viewpoint of numerical error.\(^1\) In this approach, the entire analysis zone is divided into a liquid and a solidified material (S.M.) zone. For the liquid zone, multi-dimensional numerical simulation is necessary for most actual process analysis because of complex geometry and variable material properties. However, for S.M. analysis, a simpler approach can be applied under the following assumptions: (1) lateral heat flux is negligible. (2) solid flow is steady. (3) solid deformation is negligibly small as to the heat transfer analysis. These assumptions are valid for a variety of types of C.C., e.g. thin slab casting, twin roll casting, melt spinning, and EFG. Moreover, these assumptions transform the original multi-dimensional steady conduction problem into the one-dimensional unsteady conduction problem with a moving boundary, i.e. the Stefan problem.\(^2\)\(^-\)\(^5\)

As explained later, in the coupling methods between the S.M. zone and the liquid zone, weak coupling by means of a set of given \( v_m \) and feeding \( q''_{s,m} \) back in the solid-Stefan problem is most appropriate. This Stefan problem could be classified as the Inverse Stefan problem.\(^5\) However, this Stefan problem is actually a direct problem in which the problem is always well posed; therefore in the present paper, this Stefan problem is called the Thin-S.M. Problem (TSMP) to distinguish it from the ill-posed inverse Stefan problem.

Hence, the solution method for TSMP is discussed.

1.1 Analytical approach

For the multi-domain approach, the quasi-steady approach in the S.M. is commonly used.\(^7\)\(^,\)\(^8\) However, the quasi-steady approach causes a significantly large numerical error when Stefan number \( Ste (= \frac{Cp(T_m - T_{s,0})}{Lh}) \) is not very small (e.g. \( Ste > 1 \)).\(^5\) This is typical in the case of, e.g. rapid cooling. For unsteady governing equations, analytical solutions exist only under simple boundary conditions as in the Neumann problem,\(^2\) which is too simple a condition to express the actual casting processes.

1.2 Direct numerical approach

The direct numerical approach, in which the original governing equations are directly discretized into a deference equations system, could give a solution for general boundary conditions. However, a simple and non-iterative (i.e. explicit) solution method such as the Rung-Kutta method is not applicable to the general boundary conditions in the TSMP for general C.C. processes, in the case of, e.g., time-dependent boundary temperature \( T_{s,0}, T_m \). Therefore, an iterative (i.e. implicit) direct numerical approach, e.g. the finite difference method (FDM),\(^9\)\(^,\)\(^10\) the finite element method (FEM),\(^11\) or the boundary element method (BEM),\(^12\) is necessary for the TSMP. For this approach, a significantly large calculation cost is inevitable because a rather small time increment compared with the required casting time is necessary; the time increment is limited by the Courant-Friedrich-Levy (CFL) condition, which governs the calculation stability and accuracy, even in an implicit method because of the special limitation for solidification analysis. This limitation is that the movement of the S.F. in a time step must not exceed the spatial grid interval, which should be sufficiently small for accuracy.\(^13\) Despite the recent development of computer technology, this method remains too expensive for an iterative coupling method between the liquid and the S.M. zone; this coupling method is essential as the solution process is stabilized especially for a potentially unstable problem such as in melt-spinning analysis.\(^14\)

1.3 Asymptotic approach

The asymptotic approach includes the classical integral method,\(^2\) the perturbation method,\(^15\)\(^-\)\(^17\) the inverse method,\(^18\)\(^-\)\(^20\) and the spectral method.\(^2\)\(^1\) In these methods, the original governing equations are approximately expressed by a sum of a series of expansion functions of \( x_2 \) and \( t \); each expansion function could be decided analytically under the given boundary conditions.

In the perturbation method, the expansion function is generally assumed to depend only on variable \( \xi (= h_1 - x_2) \) under the assumption of \( Ste \ll 1 \),\(^15\)\(^-\)\(^19\) temperature distribu-
tion $T_i[\xi]$ was expressed as the $n$-th power of $Ste$. Because of
the nature of a power function, $n$ is limited in the second
order at maximum from the view of stability of the function
shape;\(^5\) the shape of the final approximated function could be
so simple to express the possible complex curve of $T_i$
distribution that is caused with, for example, the existence of
impinging liquid flow on a specific area of the S.F. (this
generally occurs in the C.C.). Moreover, the assumption
$Ste \ll 1$ could be invalid for a rather rapid solidification.

Grzymkowski and Slota\(^{20}\) proposed an inverse method in
which the expansion function is expressed with independent
variables $x_2$ and $t$ under the given history of $h_s$; the method
is applicable to the wider range of the boundary conditions.
However, this method is limited in the assumption of $Ste \ll 1$
to decompose the expansion function. Fredrick and Greif\(^{19}\)
proposed the inverse method for a general magnitude of $Ste$.
However, in this method, the expansion function is based on
the Taylor expansion on the S.F.; the pre-defined history of $h_s$
is limited to a rather simple shape. In addition, in the most
inverse Stefan problem, both the temperature and the heat
flux on the S.F. have to be given in advance in order to
acquire the field values on the free boundary (i.e. a boundary
where no boundary conditions are given). On the contrary,
the heat flux on the S.F. should be a solution variable in the
TSMP; the problem to be solved is totally different regarding
the inverse problem and the TSMP.

The spectral method was applied to some Stefan problems
(but not to TSMP) with Fourier-expansion or Chebyshev-
polynomial approximation.\(^{22,23}\) These studies suggest their
applicability to the general boundary condition. However, the
conventional method requires significant cost; implicit solu-
tion method and/or tolerable numerical integrations for each
basis function are necessary. In the conventional method,
application to the variable thermal diffusivity is generally
difficult. Moreover, by definition, the problem setting in these
works differs from the TSMP.

Therefore, none of the above conventional methods is
appropriate for solving the TSMP in terms of accuracy or cost
efficiency.

In the present paper, a non-iterative (explicit) asymptotic
solution method for TSMP based on the spectral method
provided with a series of specialized basis function is
described. This method is applicable to the general boundary
condition and variable material properties even in the case of
$Ste > 1$.

2. Nomenclature

$q_i'$: heat flux
$q_i''$: steady component heat flux

$Ste$: Stefan number
$T$: temperature

$Ta, dT$: average/deviation component of $T$

$dT$': basis function of $dT$

$dTf$': component of $dT$

$t$: time

$\Delta t$: time step

$t_1': t - t_{i-1} + \Delta t_i/2$

$u$: velocity

$U_i$: pull velocity in C.C.

$v_{A}$: $-v_{oa}$

$\nu$: solidification rate

$x$: coordinate value

$\alpha$: thermal diffusivity

$\beta_1, \beta_2$: model constant

$\epsilon$: emissivity

$\sigma$: Boltzmann’s factor

$\theta$: deviation temperature

$\theta_0, \theta_1, \theta_2$: component of $\theta$

$\mu$: viscosity

$\xi$: relative coordinate

$\rho$: density

(Subscripts)

$\theta$: C.D.-S.M. interface

$fmr, lat$: former/latter half of time step

in: inlet

ini: initial

l: liquid

i: time step index

j: basis function index

m: the S.F. or index m

out: outlet

r: C.D. (roll)

ref: reference

s: S.M.

1, 2: coordinate direction

$B_1$, $B_2$: direction along/vertical to the roll surface

(Superscripts)

$\omega$: $x_2$ of discretized point of $dT$

3. Mathematical Model

3.1 Liquid-S.M. coupling for TSMP

The strong coupling method (the field values in the liquid
and the solid zone are solved simultaneously as a simulta-
neous equation system) and the weak coupling method (the
field values are solved individually and a coupling method is
applied to the zone boundary) are candidate methods, but in
this paper, the weak coupling method was adopted because
strong coupling is significantly expensive.

3.1.1 Quasi-static fluid case

The TSMP could be reduced to an unsteady one-dimen-
sional problem; the solution for the conventional Stefan
Problem is applicable. Thus, no further discussion is required
here.

3.1.2 Convective fluid in the case of $Ste \ll 1$

The heat transfer rate distributes on the S.F. by means of
fluid flow. The solidification rate and $h_s$ are so small compare
with the liquid flow speed and the representative flow length scale that the liquid flow field is almost irrelevant to the S.M. behavior except for the heat boundary condition. Thus, the one-way coupling approach, which implies relatively small calculation cost on TSMP, from the liquid zone to the S.M. zone is applicable. The following present solution method is applicable to this case as well as the conventional method.

3.1.3 Convective fluid in the case of Ste in general magnitude

A multi-dimensional liquid flow is affected by the S.F. distribution; thus, the two-way coupling method between the liquid and the S.M. zone is necessary. For this case, the following present solution method is appropriate because of the favorable balance of accuracy and cost; the cost is important in the iterative (implicit) solution approach between the calculation zones.

Hence, the favorable coupling boundary values in the weak coupling method are described. In zone coupling, all but one boundary value out of four variables, i.e. $T_m$, $v_m$, $q''_{slm}$, and $q''_{srm}$ should be given or dependent on the other boundary values in each zone so as to make the problem well posed. The remaining variable becomes a solution variable, which must not be common between the two zones. In the boundary variables, $v_m$, $q'_{slm}$, and $q'_{srm}$ are dependent on each other by means of the following equation.

$$\rho_Lh v_m = q'_{slm} - q'_{srm} \quad (1)$$

In sub-cooled solidification, $T_m$ could be expressed as a function of $v_m$ and the local material composition. The solidification rate $v_m$, which expresses the zone boundary shape explicitly, should naturally be a solution variable for either zone. This is because the flow field, which governs the heat transfer on the S.F., is highly sensitive to the boundary shape in the multi-dimensional numerical analysis for the liquid zone. On the other hand, in the case of, for example, $Ste \gg 1$ (i.e. $Lh/q''_{slm} \to 0$), $v_m$ could become highly unstable when $v_m$ is not a solution variable but is calculated from the other variables with eq. (1). Thus, $v_m$ should be a solution variable. As for $T_m$, because this variable generally varies in a smaller range during the calculation process than the other variables, $T_m$ could be acquired explicitly by means of the function of $v_m$ as well as by means of an implicit simultaneous solution method for $v_m$ and $T_m$. As for $q''_{srm}$ and $q''_{slm}$, these variables exist only in the zone to which they belong respectively; i.e. $q''_{srm}$ is for the liquid zone, $q''_{slm}$ is for the S.M. zone. Thus, two sets of boundary variable couplings are possible for the S.M. zone: (1) $q''_{srm}$ is given from the liquid zone and $v_m$ is fed back to it. (2) $v_m$ is given from the liquid zone and $q''_{slm}$ is fed back to it, that is the TSMP. Although the former method is an orthodox method in the case of $Ste \ll 1.0$, the latter was adopted in this paper for the following reason. In the former coupling, the iterative calculation process could become unstable because the zone interior heat is transported only diffusively in the Stefan problem. This limitation of the heat transfer route may cause problems. For example, $v_m$ becomes infinite when given $q''_{srm}$ tends to 0, or $h_1$ become close to 0 (under negative $v_m$) when the given $q''_{slm}$ is a relatively large value; this situation commonly occurs during the calculation process. On the contrary, in the latter coupling, the iterative calculation process is more stable because any given $v_m$ returns a finite $q''_{slm}$. In this coupling, the potential calculation instability is transported from the S.M. zone to the liquid zone calculation. However, the calculation process in the liquid zone is generally stable for input $q''_{slm}$, which is calculated from the given $q''_{srm}$, because the advective heat transfer in the liquid zone helps the problem to satisfy the given boundary conditions; for example, any high given $q''_{slm}$ result in a finite $v_m$ output, or a negative $v_m$ in a rather small magnitude become an output for the case in which the given $q''_{slm}$ becomes zero. Therefore, the TSMP is the most appropriate for the boundary variable coupling.

3.2 Liquid model

In the TSMP, $h_1$ is so thin that the actual shape of the S.F. could be approximated with locally variable exit velocity $v_A$ on the imaginary flat shape S.F.. This approximation reduces the analytical cost because exploring the S.F. position is unnecessary. The boundary conditions on the S.F. become:

$$u_1 = \left( \rho_s/\rho_l \right) U_A \quad (2)$$

$$u_2 = v_A (\rho_s/\rho_l) = -v_m (\rho_s/\rho_l) \quad (3)$$

$$T = T_m \quad (4)$$

As mentioned previously, $q''_{slm}$ is given and $v_A$ is a solution variable.

3.3 S.M. model

The present solution method for the TSMP is described below.

3.3.1 Governing equation

The governing equation is:

$$\partial T_s/\partial t = \partial /\partial x_2 (\alpha_s \partial T_s /\partial x_2) \quad (5)$$

$$T_s = T_m, \quad x_2 = h_1 \quad (6)$$

$$T_s = T_0, \quad x_2 = 0 \quad (7)$$

$$h_1 = \int_0^t v_m \, dt \quad (8)$$

Here, $T_s$ is divided into steady component $T_{s0}$ and deviation component $dT_s$ assuming a step of change of $h_1$, $T_{s,m}$, and $T_{s,0}$ at the middle of the time step (Fig. 1). The physical meaning of this decomposition is that the given temperature distribution at the middle of the time step should become that for steady state $T_{s0}$ along the time advance when the boundary
conditions $h_s$, $T_m$, and $T_s,0$ are kept in the rest time step; $dT_s$ correspond to the decaying component. Thus, eq. (5) becomes:

$$\frac{\partial(T_s + dT_s)}{\partial t} = \frac{\partial}{\partial x_2}[a_i[T_s][\partial(T_s + dT_s)/\partial x_2]] \quad (9)$$

Assuming $T_s \gg dT_s$, which is plausible for most C.C. processes, let $a_i[T_s]$ be $a_i[T_s,0]$; we can divided eq. (9) into the following the steady component and the deviation component equation.

$$d/\partial x_2(a_i[T_s]dT_s/\partial x_2) = 0 \quad (10)$$

$$T_s = T_m, \quad x_2 = h_s \quad (11)$$

$$T_s = T_s,0, \quad x_2 = 0 \quad (12)$$

$$\frac{\partial dT_s}{\partial t} = \frac{\partial}{\partial x_2}(a_i[T_s]dT_s/\partial x_2) \quad (13)$$

$$dT_s = 0, \quad x_2 = h_i \quad (14)$$

$$dT_s = 0, \quad x_2 = 0 \quad (15)$$

Here, $x_2$ represents the solidification progressing direction. The solution of the heat flux on the S.F. in the $i$-th time step, $q_{s,i}''$, is acquired with the sum of the heat flux for the steady component $q_{s,i}^{*''}$, and the deviation component $dq_{s,i}''$. The surface heat flux on the cooling device $q_{s,i}''$, is also acquired in the same way.

### 3.3.2 Steady component equation

The solution for eq. (10) is expressed only with the integral of $1/a_i$. Thus, many functions for $a_i$ could easily be adopted so as to acquire an analytical solution. In the case of, for example, $a_i$ is constant, $T_s$ becomes:

$$T_s = (T_m - T_s,0)(x_2/h_s) + T_s,0 \quad (16)$$

In the case of $a_i = a_i[T_s] + a_2$ (thermal properties are often described in this form of function in material handbooks), $T_s$ becomes:

$$T_s = \left(-a_2 + \left(\frac{a_2}{h_s} + \frac{x_2}{a_2}T_m(a_2T_m + 2a_2) + \left(1 - \frac{x_2}{h_s}\right)a_2T_s(0) + 2a_2\right)\right)^{1/2} \quad (17)$$

The heat flux in the steady component $q_{s,i}^{*''}$, and $q_{s,i,0}^{*''}$ is readily acquired analytically with differentiation of eq. (16) or (17) at $x_2 = h_i$, or $x_2 = 0$.

### 3.3.3 Deviation component equation

The collocation spectral method is applied to solve eq. (13); in this method, the solution is expressed by means of the sum of a series of basis functions. The author adopted the following basis function $dT_{i,j}(i$ represents the $i$-th time step, $j$ represents the $j$-th basis function). In time step $i$, the basis functions are initialized at the middle of the time step just after the above-mentioned step change of $h_s$ and $T_s$. The initial value of the basis function is:

$$dT_{i,0,j} = (dT_{i-1,j}^0/x_{i,0,j}) \quad x_2 = b_{i,j,0} \quad (18)$$

$$dT_{i,j,0} = (dT_{i,j}^0(h_{i,j} - x_{i,0,j})) \quad (h_{i,j} - x_2)$$

$$= b_{i,j}(h_{i,j} - x_2), \quad x_{i,j} < l_{x_2} \quad (19)$$

The basis function is Fourier expanded; the $m$-th expanded term for the $j$-th basis function, $dT_{i,j,m,ini}$ is:

$$dT_{i,j,m,ini} = \left(\frac{2}{h_{i,j}}\right) \int_{0}^{x_{i,j}} f_{m,j,ini}\sin\left(\frac{m\pi x_2}{h_{i,j}}\right) dx_2$$

$$= \left(\frac{2}{(m\pi)^2}\gamma_{i,j}(1 - \gamma_{i,j})\sin[m\pi x_2/h_{i,j}]\sin[m\pi x_2/h_{i,j}]\right)$$

Here, $y_{i,j} = x_{i,0,j}/h_{i,j}$. The temperature distribution $dT_{i,j,m,ini}$ always satisfies eq. (14) and (15); therefore, the solution of eq. (13) for the original initial condition could be expressed with the sum of the solution of eq. (13), $dT_{i,j,m,ini}$, for each initial condition $dT_{i,j,m,ini}$. Under constant $a_{i,j}$ for each $dT_{i,j,m,ini}$, the solution of eq. (13) becomes as follows in the period of $0 \leq t' \leq \Delta t/2, \quad t' = t - t_{i,j-1} + \Delta t/2$.

$$dT_{i,j,m} = \left(\frac{2}{(m\pi)^2}\gamma_{i,j}(1 - \gamma_{i,j})\sin[m\pi x_2/h_{i,j}]\exp\left[-\left(\frac{m\pi}{h_{i,j}}\right)^2 \alpha_{i,j,t}\right]\right)$$

The partial differentiation of $dT_{i,j,m}$ analytically becomes:

$$\frac{\partial dT_{i,j,m}}{\partial x_2} = \left(\frac{2}{(m\pi)^2}\gamma_{i,j}(1 - \gamma_{i,j})\cos[m\pi x_2/h_{i,j}]\exp\left[-\left(\frac{m\pi}{h_{i,j}}\right)^2 \alpha_{i,j,t}\right]\right)$$

In the spectral method, the deviation heat flux on the S.F. for the latter half of the $i$-th time step ($t_{i,j-1} + \Delta t/2 \leq t \leq t_i$), $dq_{s,j,i,lat}$, is approximated with the sum of the basis function $dT_{i,j}$ differentiation, which is expressed with the sum of $dT_{i,j,m}$ differentiation.
When \( N \) is significantly small, \( M \) could be a huge value. However, in this condition, the step change of \( h_i \) in a time step is so small that it hardly affects the temperature field; the calculation could be performed so that no change of \( h_i \) occurs during the time step. Therefore, the basis function of the last time step could be carried over; no limitation emerges for \( M \). Thus, \( M \) could be always a finite value.

The thermal diffusivity for the \( j \)-th basis function in the \( i \)-th time, \( \alpha_{x,i,j} \), could be set:

\[
\alpha_{x,i,j}[T] = \alpha_i[Ta(x_{s,i,j})]
\]  

Equation (28) generally causes a numerical error. However, because of its function shape, the deviation temperature at \( x_{s,i,j} \), \( dT_{i}[x_{s,i,j}] \), tends to be most affected by its component \( dT_{i}[x_{s,i,j}] \), which gives a plausible \( \alpha_i \) at \( x_{s,i,j} \). Thus, the numerical error was relatively small as described in the following chapter.

To decompose the initial deviation temperature \( dT_{i}[x_{s,i}, t=0] \) into the \( N \) initial basis functions \( dT_{i,j,m} \), \( N \) simultaneous linear equations should be solved so that the sum of \( dT_{i,j,m}[x_{s,i,j}] \) \((j = 1, N)\) agrees with the given \( dT_{i}[x_{s,i,j}, t=0] \), which is the collocation method. \(^{21}\) The inverse matrix for solving the simultaneous equations only depends on \( \gamma_{i,j} \); the given \( dT_i[x_{s,i,j}, t=0] \) fixes the basis functions.

### 3.4 Cooling device (C.D.) model

The C.D.-Model is modeled after the heat transfer at the surface (the other side of the liquid side) of S.M. by means of giving the surface temperature. The C.D.-Model could be divided into two cases with the magnitude of the Biot number Bi (= \( \frac{h_i}{k_i} \) or \( \frac{h_i}{H R_k} \)); that is the Surface-Heat-Resistance Dominating Case \((Bi < 1)\) and the Interior-Heat-Resistance Dominating Case \((Bi > 1)\).

#### 3.4.1 Surface-heat-resistance dominating case

This is typical in the case of an open casting of ribbon such as EFG. The surfaces of a ribbon are cooled relatively weakly with the radiation and the heat advection by atmosphere. The inner heat resistance \( h_i/k_i \) is relatively small. Thus, the heat resistance at the surfaces is dominant in total heat transfer. Consequently, the initial surface temperature in casting is close to the melting point. The surface temperature gradually reduces in a wide range to the atmospheric temperature during casting.

The boundary condition at the surface is:

\[
-k_i(\frac{\partial T_{s,0}}{\partial x_2}) = Hi(T_{s,0} - T_{a}) + \varepsilon\sigma(T_{s,0}^4 - T_{a}^4)
\]

In this boundary condition, surface temperature \( T_{s,0} \) is defined only implicitly. Therefore, \( T_{s,0} \) should be decided iteratively.

#### 3.4.2 Interior-heat-resistance dominating case

This is typical in the case of the other rather ordinary C.C. processes, where C.D. (e.g. a mold, a cooling roll) is usually designed to be composed with highly heat conductive materials for efficient cooling. Most of these C.D. are also accompanied by massive water cooling on their back. To discuss the interior heat resistance, supposing a steady heat transfer for the Stefan problem, the heat flux through the S.M. \( q'' \), is:
In most C.C. processes, the heat resistance at the surface \( H_r \) is designed to be smaller than inner heat resistance \( h_i/k_i \) except for at very early solidification (i.e. at very small \( h_i \)). Therefore, in most areas, the interior heat resistance is dominant in the total heat transfer.

Because the actual processes correspond to the unsteady Stefan problem, the actual temperature distribution in the S.M. has to be calculated not with eq. (30) but with the S.M.-Model. However, eq. (31) could be valid in the actual processes because the thickness of the heat resistance layer at the C.D. surface is so thin that the local quasi-steady state is expected there.

When the contact between S.M. and C.D. is perfect (i.e. \( H_r = 0 \)) and the surface temperature of the C.D. is regarded as constant value \( T_{r,0} \), the surface temperature of S.M. \( T_{r,0} \) is equal to \( T_{r,0} \). Otherwise, \( T_{r,0} \) should be decided iteratively with the combination of eq. (31), the S.M.-Model, and the C.D. Model (i.e. Implicit-Coupling method) in general. However, the following plausible assumptions in the C.C. could allow escape from the expensive Implicit-Coupling method to a simple Explicit-Coupling method. The assumptions are: (1) \( T_{r,0} - T_{r,ini} \ll T_m - T_{r,0} \) (2) \( (\partial T_{r,0}/\partial t) \Delta t \ll T_m - T_{r,0} \). The calculation steps of the Explicit-Coupling method are: (1) At a certain time step \( i \), the surface temperature of the C.D. \( T_{r,0} \) is calculated with the C.D.-Model, which is explained later. The value of the surface heat flux of the S.M. at the previous time step \( q^r_{s,0,i-1} \) is used as the heat boundary condition at the C.D. surface. (2) The surface temperature of the S.M. at time step \( i \) \( T_{s,0,i} \) is calculated with eq. (31) on the condition of \( q^r_{s} = q^r_{s,0,i-1} \). (3) The temperature distribution in the S.M. is calculated with the S.M.-Model. This temperature distribution gives \( q^r_{s} \) for the next time step calculation.

The C.D.-Model gives the instantaneous surface temperature of the C.D. from the given history of the surface heat flux by way of solving the temperature distribution in the C.D.. In the relevant calculation area, the heat transfer analysis reduces to a boundary value problem for heat conduction in a fixed calculation area. Thus, there are analytical solutions for typical analysis cases. Here, for example, the solution for melt spinning is described.

The material properties of C.D. are set constant because of its relatively less changeable temperature. The C.D. is regarded as a semi-infinite material and the initial temperature is uniform. As the actual melt spinning process condition, these assumptions are plausible. The governing equation is as follows.

\[
\begin{align*}
q''_s &= (T_m - T_{r,0})/(H_r + h/k_i) \\
q''_r &= (T_{r,0} - T_{r,ini})/H_r
\end{align*}
\]  

(30) (31)

Because of the equation linearity, the temperature distribution at time step \( i \) could be composed with superposition of the response deviation-temperature distributions to the rectangular-heat-input at every past time step. Each response deviation-temperature distribution is calculated with a superposition of the response deviation-temperature distributions to two step-heat-input whose beginning time differs. For a semi-infinite body, the deviation-temperature distribution of response to a step-heat-input that begins at time \( t_j \), \( \theta_j \), under the conditions of \( \theta_{r,ini} = 0 \), \( \theta_j \), \( \theta_j \), \( \theta_j \), \( q''_{r,0} = 0 \) \( (t < t_j) \), and \( q''_{r,0} = q''_{r,0,j} (t \geq t_j) \) is as follows.\(^3\)

\[
\begin{align*}
\theta_0[t] &= \frac{1}{k_i} \left\{ 2 \sqrt{\frac{\alpha_t}{\pi}} \exp \left[ -\frac{r^2}{2(\alpha_t t)} \right] \right. \\
& \quad \left. - x_2 \operatorname{erfc} \left[ \frac{x_2}{2\sqrt{\alpha_t t}} \right] \right\}
\end{align*}
\]  

(34)

The deviation-temperature distribution of the response to a rectangular-heat-input, \( \theta_2 \), under conditions of \( q''_{r,0} = 0 \) \( (t < t_j, t > t_{j+1}) \) and \( q''_{r,0} = q''_{r,0,j} (t_{j+1} \geq t \geq t_j) \) is:

\[
\begin{align*}
\theta_2 &= q''_{r,0,j} [00[t - t_j] - 00[t - t_j - \Delta t]]
\end{align*}
\]  

(35)

Thus, when each time step is constant (i.e. \( \Delta t = \Delta t \)), the deviation-temperature distribution at time \( t_i \) \( (= i\Delta t) \) becomes:

\[
\begin{align*}
\theta_{r,i} &= \sum_{j=1}^{i} \theta_2[t_i - t_j] \\
&= \sum_{j=1}^{i} q''_{r,0,j} [00[i - j + 1]\Delta t] \\
& \quad - \sum_{j=1}^{i-1} q''_{r,0,j} [00[i - j]\Delta t] \\
&= q''_{r,0,i} [00[i\Delta t]] + \sum_{j=1}^{i-1} (q''_{r,0,j+1} - q''_{r,0,j}) [00[i - j]\Delta t]
\end{align*}
\]  

(36)

Surface temperature \( T_{r,0} \) is defined at \( x_2 = 0 \) in eq. (36). Consequently, \( T_{r,0} \) becomes:

\[
T_{r,0} = \frac{2}{k_i} \sqrt{\frac{\alpha_t \Delta t}{\pi}} \times \left\{ q''_{r,0,1} \sqrt{i} + \sum_{j=1}^{i-1} (q''_{r,0,j+1} - q''_{r,0,j}) \sqrt{i - j} \right\} + T_{r,ini}
\]  

(37)

3.5 Advantage of the present solution method

The present method gives the solution temperature field explicitly without the limitation of time increment. In addition, this method requires less computer memory because of the nature of the one-dimensional method and less limitation for the spatial interval; the limitation occurs only on the S.F.

The present model analytically gives the temperature differentiation, with which the heat flux is expressed, at any point of \( x_2 \). On the contrary, in the finite difference methods, the differentiation is only acquired with the difference approximation; thus, a micro spatial interval is necessary to achieve accuracy. The present method is also applicable to the Stefan problem in \( Ste \gg 1 \); most of the analytical methods and the perturbation method are incapable of handling this condition. Moreover, the present model considers general boundary conditions and \( \alpha_t \) variation by means of a multi-series of the decomposed functions (each series corresponds to the basis function, which could be given.
different material properties); on the contrary, the conventional explicit spectral method is weak at the $\alpha$ variation because of its approximation by means of only a single series of the decomposed functions (each decomposed function corresponds to the basis function). These advantages are due to the shape of the present basis function for the spectral method. Therefore, the present method is advantageous in the balance of cost and accuracy.

4. Validation

4.1 Schwarz-like problem

A simple and typical Stefan problem, which is similar to the well-known Schwarz Problem,\textsuperscript{23} was solved compared with the conventional method.

4.1.1 Problem description

The one-dimensional unsteady heat conduction problem between a roll and melt (liquid and S.M.) was solved under the three conditions of Table 1 ($x_3$ direction). The origin is the interface between the melt and the C.D.). These conditions correspond to actual casting processes as ferroalloy casting with a copper C.D., in which $Ste$ could become the largest. The intention of the validations for these cases is as follows: (1) CASE 1: the applicability to the infinite large $Ste$ ($Lh = 0$ means an amorphous casting). (2) CASE 2: the applicability to the highly variable heat conductivity. (3) CASE 3: the applicability to the large $Hr$ and $Lh$. As the initial condition, the temperature for both the melt and the C.D. are set according to a conventional method; That is the finite volume method (FVM),\textsuperscript{25} which is the most popular computational fluid dynamics model for engineering purposes. The FVM was accompanied by the enthalpy method\textsuperscript{13} for the solidification model. The mesh number 4000 and the time increment $5 \times 10^{-8}$ s were adopted so as to make the integral of $q_m$ independent from the discretizing parameters. The liquidus temperature was set 2 K above the solidus temperature ($T_m$) for calculation stability. The calculated $T_m$ position-transition was applied to the present method as the pre-defined $h_r$.

4.1.2 Results and discussion

The present temperature distributions agree with the conventional method within 5% (as a local value) for all cases (Fig. 2). The agreement for the heat fluxes is within 10% (as a local value) and 2% (as a time integral value) (Fig. 3). These accuracies are sufficient for ordinary engineering purposes. As for calculation time, the conventional method took 8 min for the casting term of $10^{-4}$ s, whereas only about 1 second is necessary for the present method; in both calculations, an Intel® Xeon® 3 GHz CPU\textsuperscript{26} was used. Therefore, the present method is accurate and cost effective even in the case of $Ste \gg 1$ or variable material properties.

4.2 Twin roll casting

A more practical problem was solved compared with the experimental data in the twin roll casting of paraffin performed by Shiomi & Osakada.\textsuperscript{27}

4.2.1 Problem description

The geometry and material properties follow those of the

---

**Table 1 Calculation conditions.**

<table>
<thead>
<tr>
<th></th>
<th>CASE 1</th>
<th>CASE 2</th>
<th>CASE 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_{ref}$ /K</td>
<td>300</td>
<td>1573</td>
<td>1023</td>
</tr>
<tr>
<td>$T_{ref}$ /K</td>
<td>1000</td>
<td>1423</td>
<td>1498</td>
</tr>
<tr>
<td>$x_{2,ref}$ /μm</td>
<td>30</td>
<td>100</td>
<td>10000</td>
</tr>
<tr>
<td>$k_0$ /W/m·K$^{-1}$</td>
<td>400</td>
<td>150</td>
<td>400</td>
</tr>
<tr>
<td>$k_r$ /W/m·K$^{-1}$</td>
<td>25</td>
<td>0.019 × $T_r$ + 0.1</td>
<td>30</td>
</tr>
<tr>
<td>$\rho_0$ /kg/m$^3$</td>
<td>8700</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\rho_r$ /kg/m$^3$</td>
<td>6700</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$Cp_0$ /J/kg·K$^{-1}$</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>$Cp_r$ /J/kg·K$^{-1}$</td>
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<td></td>
</tr>
<tr>
<td>$Lh$ /J/kg</td>
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<td>$2.3 \times 10^7$</td>
<td>0</td>
</tr>
<tr>
<td>$Hr$ /mK/W</td>
<td>0</td>
<td></td>
<td>$1.0 \times 10^{-7}$</td>
</tr>
<tr>
<td>$Ste$</td>
<td>$\infty$</td>
<td>($\sim 1$)</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

---

**Fig. 2 Temperature Distribution in the $x_3$ direction.** (a) CASE 1, (b) CASE 2, (c) CASE 3.
experiment. The cooling roll rotates at peripheral speed $u_r$ 0.012 m/s in the outlet direction. To accommodate the problem to the present method, the entire calculation zone is divided into the liquid and S.M. zone at the S.F. (solidus temperature $T_m$) (Fig. 4). The following assumptions are imposed; (1) the curvatures of the roll and the S.F. are ignored in the S.M. zone. (2) the curvature variation of the roll surface is regarded as the Stefan problem. The recirculation flow transfers the cooled mush from around the outlet to the inlet direction; this phenomenon was clearly observed in some experimental cases. (3) the solidification is ignored in the S.M. zone. (4) no deformation in the S.M. occurs. (5) the heat transfer in the S.M. is regarded as the Stefan problem. Because the liquid is so thin ($x_B$ direction), or vertical (the $x_H$ direction) there that the curvature of the roll surface is rapidly increased around the outlet; this is because the liquid is so thin ($x_B$ direction) the temperature of the main stream more rapidly decreases in the stream ($x_B$) direction than in the up-Stream. Therefore, the simplification derived from the assumption of, for example, the static flow field or a classical boundary layer theory (these assumptions were common in the conventional Stefan problem approaches) is inadequate.

4.2.2 Results and discussion

Despite the rather simple geometry, the calculated velocity field (Fig. 5) and the shape of the S.F. curve (Fig. 6) are complicated because of the existence of a recirculation flow in the liquid zone. The recirculation flow transfers the cooled mush from around the outlet to the inlet direction; this phenomenon was clearly observed in some experimental cases. (3) $h_s$ is rapidly increased around the outlet; this is because the liquid is so thin ($x_B$ direction) the temperature of the main stream more rapidly decreases in the stream ($x_B$) direction than in the upper-stream. Therefore, the simplification derived from the assumption of, for example, the static flow field or a classical boundary layer theory (these assumptions were common in the conventional Stefan problem approaches) is inadequate.

The difference of $h_s$ at the outlet between the experimental and the present result is 7%. The present method seems to express an actual casting process as well as the conventional full-2-dimensional FEM described in Ref. 28), in which the given material properties were adjusted so that the $h_s$ result is 7%. The present method seems to agree with the experimental data.

**Fig. 3** Surface heat flux transition. (a) on the S.F., (b) on the roll surface.

**Fig. 4** Schematic of the system.

**Fig. 5** Velocity vector distribution in the liquid zone.

**Fig. 6** The S.F. distribution.
However, the uncertainty analysis regarding neither the experimental data nor the material properties is described in Ref. 28). Thus, the quantitative accuracy estimation is not to be discussed for this case until at least accurate material properties for the experiment are provided.

5. Conclusion

A spectral method in which basis functions are specialized in the solidification problem was developed for the Stefan problem. In combination with the multi-dimensional CFD methods for the liquid zone, this method is adequate for the thin solidified material problem for a variety of continuous casting processes. The method is as accurate as conventional direct numerical methods but less expensive, especially in the case of $\text{Ste} \gg 1$ or the variable material properties condition.

REFERENCES

26) Intel and Xeon are registered trademarks of Intel Corporation.