1. Introduction

Different models, which describe the solid/liquid (SL) interface dynamics, were developed: sharp interface\(^1,2\) spinodal decomposition\(^3,4\) phase field model\(^1\) \textit{etc.} In\(^5\) the solidification and the melting processes were assimilated to dark cnoidal oscillation modes (described by the \((-cn^2\) elliptic function\(^6\)) and to bright cnoidal oscillation modes (described by the \((+cn^2\) elliptic function\(^6\)), respectively. Then, the interface works like a nonlinear Toda lattice\(^5\) and some properties of this lattice are given. The model was subsequently extended to a slag-metallic bath interface\(^7\).

Recently\(^8\) these phenomena were related to the broader field of nonlinear dynamics. From this perspective, in the present paper the fractal characteristics of the solidification process are analyzed.

2. Mathematical Model

The model of the SL interface used in the present paper corresponds to an infinitely thin binary domain (solid + liquid). These subdomains satisfy the set of equations given in Appendix A. This set of equations, \((A1)-(A4)\), takes a simple form based on the following observations: i) The thermal transfer in the SL interface is linear, \(i.e. q_S = -c_ST_L; q_L = +c_LT_S\); \(q_0 = 0\) where \(c_S\) and \(c_L\) are two thermal constants; ii) The liquidus and solidus lines are straight, \(i.e.\) the thermal field and the solute one are proportional\(^2,10\); iii) The length scale of the solute field, \(D/|V_0|\) is generally order of magnitude smaller than the thermal length scales, \(a_S/|V_0|\) and \(a_L/|V_0|\). Under these circumstances, one does not need to integrate the diffusion equation \((A2)\) with the boundary condition \((A4)\), the problem reducing to the study of the set of equations for the thermal transfer (for details on this problem see Refs. 9, 10),

\[
\begin{align*}
\partial_t T_S &= a_S \Delta T_S - c_ST_L, \\
\partial_t T_L &= a_L \Delta T_L + c_L T_S
\end{align*}
\]

(1a,b)

or using the non-dimensional parameters

\[
\begin{align*}
\tau &= \omega_0 t, \quad \xi = k_0 x, \quad \eta = k_0 y, \\
\alpha_S &= \alpha_S k_0^2/\omega_0, \\
\alpha_L &= \alpha_L k_0^2/\omega_0, \quad \beta_S = c_s/\omega_0, \quad \beta_L = c_L/\omega_0,
\end{align*}
\]

(2a-i)

\[
\begin{align*}
\phi_S &= T_S/T_0, \quad \phi_L = T_L/T_0, \\
\partial_t \phi_S &= \alpha_S (\partial_x^2 + \partial_y^2) \phi_S - \beta_S \phi_L, \\
\partial_t \phi_L &= \alpha_L (\partial_x^2 + \partial_y^2) \phi_L + \beta_L \phi_S
\end{align*}
\]

(3a,b)

with \((\omega_0, k_0, T_0)\) some parameters characterizing SL interface (for details see Ref. 10). The equations system \((3a),(b)\) is solved using a finite differences method.\(^2,12\) For \(\alpha_S = \alpha_L = 1/3, \beta_S = 2, \beta_L = 0.2, \tau=0–2.25\) and the initial condition \(\phi_L(\xi, \eta; 0) \sim \exp[-(\xi^2 + \eta^2)]\) the numerical solutions are presented in Figs. 1(a)–(j). It results a thermal breather (two-dimensional (2D) dark soliton (for details see Ref. 8)). For \(\alpha_S = \alpha_L = 1/3, \beta_S = 1.9, \beta_L = 1.8, \tau=0–2.25\) and the initial condition \(\phi_L(\xi, \eta; 0) \sim \exp[-(\xi^2 + \eta^2)]\) the numerical solutions are presented in Figs. 2(a)–(j). It results thermal breather pairs for increased time sequences and thermal clusters for increased time sequences, respectively.

According to\(^2,8\) the thermal breather is a micro-domain of the same supercooling degree. Thus, the thermal breather can be associated to a virtual crystallization germ, the thermal breather pair to a stable crystallization germ and the thermal cluster to a crystalline grain. Indeed, as long as the thermal transfer in the SL interface is relatively low (the undercooling in the melt is low), the crystallization germ stays virtual and does not generate solidification. The increase of the thermal transfer in the SL interface by further undercooling of the melt induces stable crystallization germs. Through the continuation of the thermal transfer, in the SL interface crystalline grains are generated.

3. Fractal Dimension Dynamics of the Solidification Process

The dynamics of the fractal dimension \((D)\) (for defining the fractal dimension see Appendix B) of the ‘physical object’ described by the numerical solutions (see Figs. 2(a)–(j)) is depicted in Fig. 3. It results the followings: i) A stable
crystallization germ is nucleating from the liquid phase ($D \cong 1.69$). Its development (and implicitly the increasing of $D$) is achieved by atoms attaching from the melt on the germ surface, as mono-atomic layers of critical dimension that represents two-dimensional germs ($D \cong 1.79$). In such a context the fractal dimension ($D \cong 1.79$) represents a manifestation of the fact that the observed fractal should have emerged from 2D sheet-like objects (such as two-dimensional germs). ii) The system relaxation (and implicitly the decreasing of the fractal dimension to $D \cong 1.76$) is achieved through the SL interface behavior as a quasi-autonomous structure.

The fractal corresponding to the process described by the set of equations (3(a),(b)) is a fractal of growth (see Appendix C). It is obtained by a growing process along a direction (anisotropy direction) starting with a thermal breather. It develops through a generalized coherence (amplitude and phase correlation) generating, in the primary stages, the thermal breather pair, and in the final ones, the thermal cluster. In such a context, the SL interface anisotropy is induced by the fractal of growth as an intrinsic property of it (see Appendix C). The fractal dimension will increase with the increasing of the anisotropy (see Appendix B and C).

4. Conclusions

The main conclusions of this paper are as follows:

i) Using a set of coupled equations, the dynamics of the SL interface is analyzed;

ii) The numerical solutions of the thermal breather, the thermal breather pair and the thermal cluster type are associated to a virtual crystallization germ, a stable crystallization germ and a crystalline grain, respectively;

iii) The fractal dimension dynamics ($D$) of the ‘physical object’ described by the numerical solutions shows that the solidification is a fractal process. In such a context, the fractal dimension $D \cong 1.79$ represents a manifestation of the fact that the observed fractal should emerge from 2D sheet-like objects (such as two-dimensional germs).

Acknowledgments

This present work was supported by Scientific Programme of NATO and by the Ministry of Foreign Affairs of Greece.
Fig. 2 The solution of the (3a, b) equations for \( \alpha_S = \alpha_L = 1/3, \beta_S = 1.9, \beta_L = 1.8, \tau(a) = 0 \rightarrow \tau(j) = 2.25, \Delta \tau = 0.25 \) and the initial condition \( \phi_0(\xi, \eta, 0) \sim \exp[-(\xi^2 + \eta^2)] \).

Fig. 3 The dependence of the fractal dimension \( D \) versus \( \tau \).

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Appendix A

For studying the SL interface one needs the followings equations:9–11)

i) Heat balance in the liquid (L) and solid (S) phases

\[ \partial_t T_i = \alpha_i \Delta T_i + q_i, \quad i = L, S \quad \text{(A-1)} \]

where \( T_i, \alpha_i \) and \( q_i \) are the temperature, thermal diffusivity and the source term in the \( i \) phase, respectively.

ii) Solute balance in the liquid (L)

\[ \partial_t c = D \Delta c \quad \text{(A-2)} \]
with \( c \) and \( D \) the solute and diffusion coefficient in the melt, respectively.

iii) Thermal flux balance of the interface. At the front, the heat conservation equation relates the local generation of latent heat to the normal temperature gradients in the liquid and in the solid,

\[
[k_s \nabla T_s - k_l \nabla T_l]_\phi \cdot \mathbf{n} = LV_\phi \cdot \mathbf{n}
\]

(A.3)

where \( L \) is the latent heat per unit volume, \( V_\phi \) and \( n \) are the growth velocity and normal at the phase boundary, directed towards the liquid, and \( k_s \) and \( k_l \) the thermal conductivities. The index \( \phi \) means that the concerned quantities are evaluated at the solidification front.

iv) Solute flux balance of the interface. Solute conservation evaluated at the solidification front.

\[
[c_0 - c]_\phi V_\phi \cdot \mathbf{n} = D(\nabla c)_\phi \cdot \mathbf{n}
\]

(A.4)

where \( c_0 \) is the solute concentration in the solid.

Appendix B

Self-similar objects like a segment of a line, a square or a cube can be defined by their Euclidian dimension which is a special case of a more general fractal dimension where the power law exponent is equal to an integer number. The derivation of this concept can be seen if a segment is divided into \( N \) equal parts, each part having been scaled down by a ratio \( r = 1/N \) from the whole. Similarly a square is divided into \( N \) similar parts by scaling it down by a factor of \( r = 1/N^{1/2} \) and a cube may be scaled down into \( N \) similar parts by a ratio of \( r = 1/N^{1/3} \). Therefore a \( D \)-dimensional self-similar object can also be divided into \( N \) similar parts by a ratio of \( r = 1/N^D \). In general, for an object of \( N \) parts, each scaled down by a ratio \( r \) from the whole, \( N r^D = 1 \), defines the similarity dimension (or, if it has a non-integer value, the fractal dimension associated to the evaluated structure),

\[
D = \frac{\log(N)}{\log(1/r)}
\]

(B.1)

For example, a fractal object like the Koch curve is composed of 4 sub-segments each of which is scaled down by a factor \( 1/3 \) from its parent (Fig. 4). The fractal dimension is calculated as \( D = \log(4)/\log(3) = 1.26 \). This curve fills more space than a simple line \( (D = 1) \) but less space than a Euclidian area of the plane \( (D = 2) \). A power law equivalent expression of this would be \( a = 1/s^D \), or equivalently \( D = \log(a)/\log(1/s) \), where \( a \) equals the number of pieces and \( s \) equals the reduction factor.\(^{13}\) This power law holds for all strictly self-similar objects regardless if they are fractal or not. For a line, square or cube, \( D = 1, 2 \) and 3, respectively and equals the Euclidean and topological dimensions for these objects.\(^{13}\) For the Koch curve shown in Fig. 4, \( a = 4 \) and \( s = 1/3 \) and on the next level of magnification \( a = 16 \) and \( s = 1/9 \). The fractional part of the similarity dimension of the Koch curve, 0.2619 is exactly equal to the power law exponent. Note that the analytical results for the dimension measure of an object are always the same regardless of the iteration level. This can be seen comparing the first iteration level of the Koch curve, where the \( D = \log(4)/\log(3) = 1.26 \) and the second iteration level where \( D = \log(16)/\log(9) = 1.26 \) (Fig. 4).

In the present paper the fractal dimension was determined by using the box-counting method. The computing process is as follows: one watches the way the number of cells \( (n) \) necessary to cover the measuring structure varies with the side of these cells. In practice, one chooses a square which has to completely cover the measuring structure, then divides the side of the square, one at a time, with 2, 4, 8, \ldots \ ([s = 1/2, 1/4, 1/8, \ldots ]) and counts the cell number \( (N(s)) \) where there are elements of the measuring structure. From the set of completed measurements with the sides of the cells of sizes \( s_1, s_2, \ldots, s_n \) one verifies the dependence of \( N(s) = c \cdot s^D \) type, from which \( D \) is deduced. In the present paper we determine the fractal dimension of some images induced by the set of equations \((3(a),(b))\). These images are sequences of the solidification process. It can be seen that the fractal dimension increases from the stable crystallization germ towards the crystallization grain. This is due to the fact that for the stable crystallization the amplitudes domain is more reduced (only a few black and white shades are observed), while for the crystallization grain some supplementary shades are present. Since these shades are measuring at the same time the anisotropy\(^{13}\) (see Appendix C), it results that with the anisotropy increasing, the fractal dimension increases too.

Appendix C

An ordered and homogenous repartition gives birth to some compact objects, which can be described by classical geometry. A more disordered repartition of the properties in 3D space (which implies anisotropy) may be described by a fractal geometry. Consequently, it is first necessary to determine a fractal object. A fractal object is generated by a recursive process which imposes a specific correlation at all scales of the object. A geometric fractal may be built starting from a way of dividing a line and repeating the same algorithm many times. In other words, it assumes an iterative application of a generating law (generator) upon an initiator (the initial state - line, surface volume). The fractals obtained by means of mathematical constructions may be classified as follows: i) fractals obtained by division (Cantor set, Koch curve, Peano curve, the Sierpinski gasket); ii) fractals obtained by growing processes (growth fractals) where one starts from a germ which develops by means of successive attaching of structures similar to the germ along some directions (the anisotropy directions). Therefore, the anisotropy appears as an intrinsic property of the fractal. In such a context, the fractal corresponding to the process described by the set of equations \((3(a),(b))\) is a growth fractal.